

THE INDEX OF REDUCIBILITY OF PARAMETER IDEALS AND MOSTLY ZERO FINITE LOCAL COHOMOLOGIES

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ABSTRACT. In this paper we prove that if M is a finitely-generated module of dimension d with finite local cohomologies over a Noetherian local ring (A, \mathfrak{m}) , and if $H_{\mathfrak{m}}^i(M) = 0$ except possibly for $i \in \{0, r, d\}$ with some $0 \leq r \leq d$, then there exists an integer ℓ such that every parameter ideal for M contained in \mathfrak{m}^ℓ has the same index of reducibility. This theorem generalizes earlier work of the second author and is closely related to recent work of Goto-Suzuki and Goto-Sakurai; Goto-Sakurai have supplied an answer of *yes* in case M is Buchsbaum.

1. INTRODUCTION

Throughout this paper, A denotes a Noetherian local ring with maximal ideal \mathfrak{m} and residue field $k = A/\mathfrak{m}$, and M is a finitely generated A -module of dimension d . In the same 1921 paper where she introduced the fundamental notion of primary decomposition of ideals, E. Noether showed that every ideal in a Noetherian ring may be expressed as a finite intersection of irreducible ideals. She also showed that if the intersection is irredundant, then the number of irreducible ideals appearing in the intersection depends only on the ideal and not on the particular intersection. This result readily generalizes to Noetherian modules. For a submodule N of M , we refer to the number of irreducible submodules appearing in any irredundant expression of N as an intersection of irreducible submodules as the *index of reducibility* of N .

By a *parameter ideal* for M we mean an ideal \mathfrak{q} that can be generated by d elements and such that $M/\mathfrak{q}M$ has finite length. We define the *index of reducibility of \mathfrak{q} on M* to be the index of reducibility of the submodule $\mathfrak{q}M$; we denote the index of reducibility of \mathfrak{q} on M by $N_A(\mathfrak{q}; M)$. Since $M/\mathfrak{q}M$ has finite length, it is known that the index of reducibility of a parameter ideal \mathfrak{q} on M is given by the *socle dimension* of $M/\mathfrak{q}M$; that is, $N_A(\mathfrak{q}; M) = \dim_k \operatorname{Hom}_A(k, M)$. (The socle of M , which we denote by $\operatorname{Soc}(M)$, is generally defined as the sum of all the simple submodules of M ; in our setting, it equals $(0 :_M \mathfrak{m})$ and is isomorphic to $\operatorname{Hom}_A(k, M)$.)

In 1956, D. G. Northcott showed that the index of reducibility of a parameter ideal in a Cohen-Macaulay local ring depends only on the ring and not on the parameter ideal [N, Theorem 3]. This result extends to modules, and the common index of reducibility of parameter ideals on a Cohen-Macaulay module is called the *type* of the module. Regardless of whether M is Cohen-Macaulay, the type is defined to be $\dim_k \operatorname{Ext}_A^t(k, M)$, where t is the depth of M .

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Although Northcott and D.G. Rees proved in 1956 that if every parameter ideal of a Noetherian local ring is irreducible then the ring is Cohen-Macaulay [NR], the constant index of reducibility of parameter ideals does not characterize Cohen-Macaulay local rings. Indeed, in 1964 S. Endo and M. Narita gave examples of non-Cohen-Macaulay Noetherian local rings having constant index of reducibility of parameter ideals [EN].

In 1984, S. Goto and N. Suzuki revived interest in the index of reducibility of parameter ideals, generalizing the examples of Endo-Narita, and paying particular attention to the supremum of the index of reducibility of parameter ideals [GSu]. We refer to this supremum as the *Goto-Suzuki type*. Although examples were given showing that the Goto-Suzuki type may be infinity, finite upper and lower bounds were provided in the case where M has finite local cohomologies [GSu, Theorem 2.1 and Theorem 2.3].

Definition 1.1. We say that M has *finite local cohomologies* if the local cohomology A -modules $H_{\mathfrak{m}}^i(M)$ have finite length for each integer i such that $0 \leq i \leq d-1$.

Implicit in [GSu] we find that if M is a module having finite local cohomologies, there exists inside every power of \mathfrak{m} a parameter ideal \mathfrak{q} such that

$$N_A(\mathfrak{q}; M) = \sum_{i=0}^d \binom{d}{i} \text{Socdim}(H_{\mathfrak{m}}^i(M)).$$

This is the lower bound for the Goto-Suzuki type mentioned above.

An equivalent condition for M to have finite local cohomologies is the existence of a standard ideal for M .

Definition 1.2. An ideal \mathfrak{a} is called a *standard ideal* for M if \mathfrak{a} is an \mathfrak{m} -primary ideal with the property that for each system of parameters x_1, \dots, x_d of M contained in \mathfrak{a} , we have

$$((x_1, \dots, x_{i-1})M :_M x_i) = ((x_1, \dots, x_{i-1})M :_M \mathfrak{a})$$

for every integer i with $1 \leq i \leq d$.

In 2003, Goto and H. Sakurai showed that if M is Buchsbaum (i.e., \mathfrak{m} is a standard ideal for M), then there exists a power of \mathfrak{m} inside which every parameter ideal for M has the same index of reducibility on M [GSaI, Corollary 3.13], necessarily equal to the lower bound of the Goto-Suzuki type. We refer to this by saying M has *eventual constant index of reducibility of parameter ideals*. In a recent paper, the second author showed that if M has dimension one, or if M has dimension two, finite local cohomologies, and positive depth, then M has eventual constant index of reducibility of parameter ideals [R, Theorem 2.3 and Theorem 3.3]. The results mentioned in this paragraph provide partial answers to the following question, which appears as [R, Question 1.2].

Question 1.3. *Suppose (A, \mathfrak{m}, k) is a Noetherian local ring having finite local cohomologies. Does A have eventual constant index of reducibility of parameter ideals?*

In this paper we provide a partial answer to this question by generalizing the results in [R]. Our main result is the following theorem.

Theorem 1.4 (Main Theorem).

Suppose M has finite local cohomologies and that $H_{\mathfrak{m}}^i(M) = 0$ except possibly for $i \in \{0, r, d\}$, where r is some integer with $0 \leq r \leq d$. There exists an integer ℓ such that for every parameter ideal \mathfrak{q} of M , if $\mathfrak{q} \subseteq \mathfrak{m}^\ell$ then the index of reducibility of \mathfrak{q} on M is independent of \mathfrak{q} and is given by

$$N_A(\mathfrak{q}; M) = \sum_{i=0}^d \binom{d}{i} \text{Socdim}(H_{\mathfrak{m}}^i(M))$$

In the process of proving this result, we provide information on the behavior of the unmixed components of parts of systems of parameters contained in a standard ideal. As a corollary to the Main Theorem, we present a result that characterizes a certain class of finite local cohomology modules as Gorenstein precisely when every power of the maximal ideal contains an irreducible parameter ideal (here we take as definition that a Gorenstein local ring is a Cohen-Macaulay local ring having an irreducible parameter ideal.)

There are some overlaps between the results of this paper and the results of the paper [GSaII] by S. Goto and H. Sakurai. Firstly, a result that is only slightly different than our Main Theorem (Theorem 1.4) appears as [GSaII, Theorem 4.2 (2)]. Secondly, a result almost identical to our Proposition 3.1 appears as [GSaII, Proposition 2.6]. While these papers were developed independently, the authors have enjoyed reading previous papers of Goto-Sakurai and Goto-Suzuki [GSaI, GSu]. The technique of studying the index of reducibility using local cohomology was learned from these papers.

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2. PROOF OF THE MAIN THEOREM AND A COROLLARY

The importance of the existence of a standard ideal is made clear by the following proposition, for which we refer the reader to [SV, Corollary 18, p. 264].

Proposition 2.1. *There is a standard ideal for M if and only if M has finite local cohomologies.*

Now we present the proof of the main theorem, Theorem 1.4; the statement and proof of several useful lemmas are postponed.

Proof. Set $W = H_{\mathfrak{m}}^0(M)$ and $\overline{M} = M/W$. According to Proposition 3.1 there is an integer n so that for every parameter ideal \mathfrak{q} for M , if $\mathfrak{q} \subseteq \mathfrak{m}^n$ then $N_A(\mathfrak{q}; M) = \text{Socdim}(M) + N_A(\mathfrak{q}; \overline{M})$. Since $H_{\mathfrak{m}}^i(\overline{M}) \cong H_{\mathfrak{m}}^i(M)$ for all $i > 0$, we may require the integer ℓ to be at least as large as n and replace M by \overline{M} for the remainder of the proof to assume that $\text{depth } M > 0$. If $H_{\mathfrak{m}}^i(M) = 0$ for all $i < d$, then we're done since M is Cohen-Macaulay. Otherwise, we have that $H_{\mathfrak{m}}^i(M) = 0$ for all i except $i = d$ and $i = r$ where $0 < r < d$.

For a submodule N of M , we denote by $U(N)$ the unmixed component of N ; that is, $U(N)$ is the intersection of the primary components of N whose associated primes have maximal dimension, equal to $\dim M/N$. According to [R, Proposition 3.2], if we let \mathfrak{a} be a standard ideal for M then there exists an integer ℓ such that $\mathfrak{m}^\ell \subseteq \mathfrak{a}$ and such that for any parameter ideal $\mathfrak{q} = (x_1, \dots, x_d)A$ for M , if

$\mathfrak{q} \subseteq \mathfrak{m}^\ell$ then the index of reducibility of \mathfrak{q} on M is given by

$$N_A(\mathfrak{q}; M) = \text{Socdim} \left(\sum_{i=1}^d \frac{U_i + \mathfrak{q}M}{\mathfrak{q}M} \right) + \text{Socdim} \left(H_m^d(M) \right).$$

where $U_i = U((x_1, \dots, \widehat{x}_i, \dots, x_d)M)$. Our task in this proof is to examine the sum appearing in the expression above.

We begin by using the assumption that $H_m^i(M) = 0$ for all integers i with $r < i < d$ to apply Proposition 3.7 and obtain

$$(2.1) \quad \sum_{i=1}^d \frac{U_i + \mathfrak{q}M}{\mathfrak{q}M} = \sum_{1 \leq i_1 < \dots < i_r \leq d} \frac{U((x_{i_1}, \dots, x_{i_r})M) + \mathfrak{q}M}{\mathfrak{q}M}.$$

If $r = d - 1$, then Equation (2.1) follows immediately from Proposition 3.7. Otherwise, because $H_m^{d-1}(M) = 0$, we may apply Proposition 3.7 to the subsystem of parameters $x_1, \dots, \widehat{x}_k, \dots, x_d$ to obtain

$$U_k + \mathfrak{q}M = \sum_{\substack{1 \leq i_1 < \dots < i_{d-2} \leq d \\ i_j \neq k}} U((x_{i_1}, \dots, x_{i_{d-2}})M) + \mathfrak{q}M,$$

for each $k = 1, 2, \dots, d$. Sum up these equations and we get

$$(2.2) \quad \sum_{i=1}^d U_i + \mathfrak{q}M = \sum_{1 \leq i_1 < \dots < i_{d-2} \leq d} U((x_{i_1}, \dots, x_{i_{d-2}})M) + \mathfrak{q}M.$$

If $r = d - 2$, this gives the equality of the numerators of Equation (2.1) and hence Equation (2.1) follows. If $r < d - 2$, then we have $H_m^{d-2}(M) = 0$ and we are able to apply Proposition 3.7 to each subsystem of parameters $x_{i_1}, \dots, x_{i_{d-2}}$ with $1 \leq i_1 < \dots < i_{d-2} \leq d$. We sum up the resulting equations (as above) and get from Equation (2.2) that

$$(2.3) \quad \sum_{i=1}^d U_i + \mathfrak{q}M = \sum_{1 \leq i_1 < \dots < i_{d-3} \leq d} U((x_{i_1}, \dots, x_{i_{d-3}})M) + \mathfrak{q}M.$$

If $r = d - 3$, Equation (2.1) follows from this. Otherwise, we continue using the assumption $H_m^\ell(M) = 0$ for the decreasing sequence of indices $\ell = d - 3, \dots, r + 1$ and applying Proposition 3.7 to all subsystems of parameters $x_{i_1}, \dots, x_{i_\ell}$ with $1 \leq i_1 < \dots < i_\ell \leq d$, and we eventually arrive at Equation (2.1).

With Equation (2.1) established, we examine one of the summands on the right side. Using Proposition 3.8 we see that

$$\frac{U((x_{i_1}, \dots, x_{i_r})M) + \mathfrak{q}M}{\mathfrak{q}M} \cong \frac{U((x_{i_1}, \dots, x_{i_r})M)}{(x_{i_1}, \dots, x_{i_r})M}.$$

Since $H_m^i(M) = 0$ for each integer i such that $0 \leq i \leq r - 1$, we use Proposition 3.13 to obtain

$$\frac{U((x_{i_1}, \dots, x_{i_r})M) + \mathfrak{q}M}{\mathfrak{q}M} \cong H_m^r(M).$$

It remains to show that the sum

$$\sum_{1 \leq i_1 < \dots < i_r \leq d} \frac{U((x_{i_1}, \dots, x_{i_r})M) + \mathfrak{q}M}{\mathfrak{q}M}$$

is direct.

By symmetry, it suffices to show that $\mathfrak{q}M$ equals the module

$$[\mathrm{U}((x_1, \dots, x_r)M) + \mathfrak{q}M] \cap \left[\sum_{\substack{1 \leq i_1 < \dots < i_r \leq d \\ i_r > r}} \mathrm{U}((x_{i_1}, \dots, x_{i_r})M) + \mathfrak{q}M \right].$$

Using basic properties of intersections, this is equivalent to showing that

$$\mathrm{U}((x_1, \dots, x_r)M) \cap \left[\sum_{\substack{1 \leq i_1 < \dots < i_r \leq d \\ i_r > r}} \mathrm{U}((x_{i_1}, \dots, x_{i_r})M) \right] \subseteq \mathfrak{q}M.$$

Using Lemma 3.4, we see that each summand $\mathrm{U}((x_{i_1}, \dots, x_{i_r})M)$, where $i_r > r$, is contained in at least one of the submodules

$$\mathrm{U}((x_1, \dots, \widehat{x}_i, \dots, x_d)M)$$

($1 \leq i \leq r$). Thus the intersection above is contained in

$$\mathrm{U}((x_1, \dots, x_r)M) \cap \left[\sum_{i=1}^r \mathrm{U}((x_1, \dots, \widehat{x}_i, \dots, x_d)M) \right].$$

According to Proposition 3.14, this intersection is contained in the submodule $(x_1, \dots, x_r)M$. This completes the proof of the theorem. \square

Corollary 2.2. *Suppose A is a Noetherian local ring having finite local cohomologies and that $H_{\mathfrak{m}}^i(A)$ is zero except possibly for $i \in \{0, r, d\}$, where r is some integer with $0 \leq r \leq d$, and $d = \dim A$. Then A is Gorenstein if and only if every power of \mathfrak{m} contains an irreducible parameter ideal.*

Proof. It is well known that if A is Gorenstein then every parameter ideal is irreducible. Now suppose that every power of \mathfrak{m} contains an irreducible parameter ideal. According to the Main Theorem, each parameter ideal inside a high enough power of the maximal ideal has index of reducibility

$$\sum_{i=0}^d \binom{d}{i} \mathrm{Socdim}(H_{\mathfrak{m}}^i(A));$$

this expression must equal 1. Since $H_{\mathfrak{m}}^d(A)$ is a nonzero Artinian module, it has a nonzero socle. Thus the socles of $H_{\mathfrak{m}}^i(A)$ ($i < d$) must be zero, so that the Artinian modules $H_{\mathfrak{m}}^i(A)$ are themselves zero. Since A has only one nonzero local cohomology module, A is Cohen-Macaulay. Since A is Cohen-Macaulay with an irreducible parameter ideal, A is Gorenstein. This completes the proof of the corollary. \square

3. SUPPORTING PROPOSITIONS

This section contains the supporting propositions used in the proof of the Main Theorem.

3.1. Reduction to positive depth.

Proposition 3.1. *Let $W = H_{\mathfrak{m}}^0(M)$. There exists an integer n such that for every parameter ideal \mathfrak{q} for M , if $\mathfrak{q} \subseteq \mathfrak{m}^n$ then the index of reducibility of \mathfrak{q} on M is given by*

$$N_A(\mathfrak{q}; M) = \text{Socdim}(M) + N_A(\mathfrak{q}; M/W).$$

Proof. Since W has finite length, there is a positive integer a such that $W \cap \mathfrak{m}^a M = 0$. Set $\overline{M} = M/W$. Since \overline{M} has positive depth, there is an \overline{M} -regular element x of A . By the Artin-Rees Lemma, there is a positive integer b such that for all positive integers c , we have $\mathfrak{m}^{b+c}\overline{M} \cap x\overline{M} = \mathfrak{m}^c(\mathfrak{m}^b\overline{M} \cap x\overline{M})$. Let $n = a + b$ and suppose \mathfrak{q} is a parameter ideal for M that is contained in \mathfrak{m}^n .

Since $\mathfrak{q}M \cap W = 0$, we have $(W + \mathfrak{q}M)/\mathfrak{q}M \cong W$, and thus an exact sequence

$$0 \longrightarrow W \longrightarrow M/\mathfrak{q}M \longrightarrow \overline{M}/\mathfrak{q}\overline{M} \longrightarrow 0.$$

Since $\text{Soc}(W) = \text{Soc}(M)$, after applying the socle functor $\text{Soc}(\ast) = (0 :_{\ast} \mathfrak{m})$ we obtain the exact sequence

$$0 \longrightarrow \text{Soc}(M) \longrightarrow \text{Soc}(M/\mathfrak{q}M) \longrightarrow \text{Soc}(\overline{M}/\mathfrak{q}\overline{M}).$$

Using the additivity of length, we see that the proof will be complete if we show that the map $\text{Soc}(M/\mathfrak{q}M) \longrightarrow \text{Soc}(\overline{M}/\mathfrak{q}\overline{M})$ is surjective.

Let $s \in M$ be a representative of a nonzero element of

$$\text{Soc}(\overline{M}/\mathfrak{q}\overline{M}) \cong ((\mathfrak{q}M + W) :_M \mathfrak{m}) / (\mathfrak{q}M + W).$$

Let \overline{s} denote the class of s in \overline{M} . Then $\overline{s} \in (\mathfrak{q}\overline{M} :_{\overline{M}} \mathfrak{m})$. Note that since

$$x(\mathfrak{m}^n\overline{M} :_{\overline{M}} \mathfrak{m}) \subseteq x(\mathfrak{m}^n\overline{M} :_{\overline{M}} x) = \mathfrak{m}^n\overline{M} \cap x\overline{M} = \mathfrak{m}^a(\mathfrak{m}^b\overline{M} \cap x\overline{M}),$$

so that $x(\mathfrak{m}^n\overline{M} :_{\overline{M}} \mathfrak{m}) \subseteq x\mathfrak{m}^a\overline{M}$. Since x is \overline{M} -regular, we have

$$(\mathfrak{m}^n\overline{M} :_{\overline{M}} \mathfrak{m}) \subseteq \mathfrak{m}^a\overline{M}.$$

Since $\mathfrak{q} \subseteq \mathfrak{m}^n$, we see that $\overline{s} \in \mathfrak{m}^a\overline{M}$, so that $s \in \mathfrak{m}^a M + W$. Replace s by another representative of \overline{s} so that we may assume $s \in \mathfrak{m}^a M$.

Now we have $\mathfrak{m}s \subseteq \mathfrak{m}^{a+1}M \cap (\mathfrak{q}M + W)$. Since $\mathfrak{q}M \subseteq \mathfrak{m}^{a+1}M$ and $\mathfrak{m}^{a+1}M \cap W = 0$, we have $\mathfrak{m}s \subseteq \mathfrak{q}M$. Thus $s \in \text{Soc}(M/\mathfrak{q}M)$. This shows that the map $\text{Soc}(M/\mathfrak{q}M) \longrightarrow \text{Soc}(\overline{M}/\mathfrak{q}\overline{M})$ is surjective, so our proof is complete. \square

3.2. The unmixed component of a sum versus the sum of the unmixed components.

Definition 3.2. Let N be a submodule of M . We say that N is *unmixed up to \mathfrak{m} -primary components* if $\mathfrak{p} \in \text{Ass } M/N$ implies that either $\mathfrak{p} = \mathfrak{m}$ or $\dim A/\mathfrak{p} = \dim M/N$.

Remark 3.3.

- (1) The unmixed component does not depend on a particular primary decomposition.
- (2) If N is a submodule of M that is unmixed up to \mathfrak{m} -primary components, then $H_{\mathfrak{m}}^0(M/N) \cong U(N)/N$.
- (3) According to [SV, Lemma 2.2, p. 71], if M has finite local cohomologies then for any part x_1, \dots, x_r ($r \geq 0$) of a system of parameters for M , the submodule $(x_1, \dots, x_r)M$ is unmixed up to \mathfrak{m} -primary components.

The following lemma explains how we usually think of the unmixed component of a submodule generated by part of a system of parameters contained in a standard ideal.

Lemma 3.4. *Suppose M has finite local cohomologies and let \mathfrak{a} be a standard ideal for M . Suppose x_1, \dots, x_r ($1 \leq r \leq d$) is part of a system of parameters for M . If $(x_1, \dots, x_r)A \subseteq \mathfrak{a}$ then*

$$U((x_1, \dots, x_{r-1})M) = ((x_1, \dots, x_{r-1})M :_M x_r).$$

Proof. Since $(x_1, \dots, x_{r-1})M$ is unmixed up to \mathfrak{m} -primary components, there is an integer n such that

$$U((x_1, \dots, x_{r-1})M) = ((x_1, \dots, x_{r-1})M :_M x_r^n).$$

This last expression equals $((x_1, \dots, x_{r-1})M :_M x_r)$, since they are both equal to $((x_1, \dots, x_{r-1})M :_M \mathfrak{a})$. \square

The following lemma is essentially a collection of several results from [SV].

Lemma 3.5. *Suppose M has finite local cohomologies and let \mathfrak{a} be a standard ideal with respect to M . Suppose x_1, \dots, x_r ($1 \leq r \leq d$) is part of a system of parameters for M . If $(x_1, \dots, x_r)A \subseteq \mathfrak{a}$, then for all integers $n_1, \dots, n_r \geq 1$ we have*

$$\begin{aligned} & ((x_1^{n_1+1}, \dots, x_r^{n_r+1})M :_M x_1^{n_1} \cdots x_r^{n_r}) \\ &= (x_1, \dots, x_r)M + \sum_{i=1}^r U((x_1, \dots, \widehat{x}_i, \dots, x_r)M). \end{aligned}$$

Proof. According to [SV, Theorem 20, Lemma 23, Lemma 24, pp 264–266], we have

$$\begin{aligned} & ((x_1^{n_1+1}, \dots, x_r^{n_r+1})M :_M x_1^{n_1} \cdots x_r^{n_r}) \\ &= (x_1, \dots, x_r)M + \sum_{i=1}^r ((x_1, \dots, \widehat{x}_i, \dots, x_r)M :_M x_i). \end{aligned}$$

We complete the proof with an application of Lemma 3.4. \square

Now we recall the connection between Koszul cohomology and local cohomology. Let I be an ideal generated by elements y_1, \dots, y_r . For a positive integer n , we use \underline{y}^n to denote the sequence y_1^n, \dots, y_r^n . We use $H^i(\underline{y}; M)$ to denote the i th cohomology module of the Koszul cocomplex $K^\bullet(\underline{y}; M)$. Since there are containments of ideals

$$(\underline{y})A \supseteq (\underline{y}^2)A \supseteq (\underline{y}^3)A \supseteq \cdots,$$

for each i the corresponding Koszul cohomology modules fit into a direct system

$$H^i(\underline{y}; M) \longrightarrow H^i(\underline{y}^2; M) \longrightarrow H^i(\underline{y}^3; M) \longrightarrow \cdots.$$

When $i = r$, the differentials are particularly simple: they are each multiplication by the product $y_1 \cdots y_r$.

For each i , it is known that the direct limit of the Koszul cohomology is isomorphic to the cohomology module $H_I^i(M)$; that is,

$$\varinjlim_n H^i(\underline{y}^n; M) \cong H_I^i(M).$$

Lemma 3.6. *Suppose M has finite local cohomologies and $H_m^r(M) = 0$ for some $1 \leq r \leq d-1$. If x_1, \dots, x_{r+1} is part of a system of parameters for M , then there exists an integer n such that*

$$((x_1, \dots, x_r)M :_M x_{r+1}) \subseteq ((x_1^{n+1}, \dots, x_r^{n+1})M :_M (x_1 \cdots x_r)^n).$$

Proof. Let $\mathfrak{q} = (x_1, \dots, x_r)A$. From [SV, Corollary 1.7, p. 29] we obtain the following commutative diagram, in which the top row is exact:

$$\begin{array}{ccccc} H^r(x_1, \dots, x_{r+1}; M) & \longrightarrow & H^0(x_{r+1}; H^r(x_1, \dots, x_r; M)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ H_{(x_1, \dots, x_{r+1})A}^r(M) & \longrightarrow & H_{x_{r+1}A}^0(H_{\mathfrak{q}}^r(M)) & & \end{array}$$

According to [SV, Lemma 22, p. 264], we have $H_{(x_1, \dots, x_{r+1})A}^r(M) \cong H_m^r(M)$, and by hypothesis this module is zero. Thus, our diagram becomes:

$$\begin{array}{ccccc} H^r(x_1, \dots, x_{r+1}; M) & \longrightarrow & H^0(x_{r+1}; H^r(x_1, \dots, x_r; M)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_{x_{r+1}A}^0(H_{\mathfrak{q}}^r(M)) & & \end{array}$$

From a simple diagram chase, we see that the map

$$H^0(x_{r+1}; H^r(x_1, \dots, x_r; M)) \longrightarrow H_{x_{r+1}A}^0(H_{\mathfrak{q}}^r(M))$$

is the zero map. Therefore the submodule

$$H^0(x_{r+1}; H^r(x_1, \dots, x_r; M)) \subseteq H^r(x_1, \dots, x_r; M)$$

is contained in the kernel K of the canonical map (obtained from the direct limit) $H^r(x_1, \dots, x_r; M) \longrightarrow H_{\mathfrak{q}}^r(M)$. Recall that $H^r(x_1, \dots, x_r; M) \cong M/\mathfrak{q}M$. It follows from the definition of the direct limit that

$$K = \frac{\bigcup_{n \geq 1} ((x_1^{n+1}, \dots, x_r^{n+1})M :_M (x_1 \cdots x_r)^n)}{\mathfrak{q}M}.$$

Furthermore,

$$H^0(x_{r+1}; H^r(x_1, \dots, x_r; M)) = \frac{(\mathfrak{q}M :_M x_{r+1})}{\mathfrak{q}M}.$$

Now we have shown that

$$\frac{(\mathfrak{q}M :_M x_{r+1})}{\mathfrak{q}M} \subseteq \frac{\bigcup_{n \geq 1} ((x_1^{n+1}, \dots, x_r^{n+1})M :_M (x_1 \cdots x_r)^n)}{\mathfrak{q}M}.$$

Since the union on the right side is a union of increasing submodules, there exists an integer n so that

$$\frac{(\mathfrak{q}M :_M x_{r+1})}{\mathfrak{q}M} \subseteq \frac{((x_1^{n+1}, \dots, x_r^{n+1})M :_M (x_1 \cdots x_r)^n)}{\mathfrak{q}M}.$$

Thus our proof is complete. \square

Proposition 3.7. *Suppose that M has finite local cohomologies and let \mathfrak{a} be a standard ideal for M . Suppose that $H_m^r(M) = 0$ for some integer r with $1 \leq r \leq d-1$. For any part of a system of parameters x_1, \dots, x_r of M , if $(x_1, \dots, x_r)A \subseteq \mathfrak{a}$, then*

$$U((x_1, \dots, x_r)M) = (x_1, \dots, x_r)M + \sum_{i=1}^r U((x_1, \dots, \hat{x}_i, \dots, x_r)M).$$

Proof. Let x_1, \dots, x_r be as in the statement of the proposition. For each integer i with $1 \leq i \leq r$, set $\mathfrak{q}_i = (x_1, \dots, \hat{x}_i, \dots, x_r)A$, where $\mathfrak{q}_i = 0$ if $r = 1$. Set $\mathfrak{q} = (x_1, \dots, x_r)A$. Choose an element x_{r+1} in \mathfrak{a} so that x_1, \dots, x_{r+1} is again part of a system of parameters for M . According to Lemma 3.5, for any integer $n > 0$ we have

$$((x_1^{n+1}, \dots, x_r^{n+1})M :_M (x_1 \cdots x_r)^n) = \mathfrak{q}M + \sum_{i=1}^r U(\mathfrak{q}_iM).$$

From this and Lemma 3.6, it follows that

$$(\mathfrak{q}M :_M x_{r+1}) \subseteq \mathfrak{q}M + \sum_{i=1}^r U(\mathfrak{q}_iM).$$

From Lemma 3.4, we have that

$$U(\mathfrak{q}M) \subseteq \mathfrak{q}M + \sum_{i=1}^r U(\mathfrak{q}_iM).$$

Using Lemma 3.4 again, we see that $U(\mathfrak{q}_iM) \subseteq U(\mathfrak{q}M)$ for each integer i with $1 \leq i \leq r$. This completes the proof. \square

3.3. The intersection of an unmixed component with a system of parameters. We thank the referee for pointing out that the following proposition is well known. Indeed, according to Trung [T, Corollary 2.6 and Proposition 3.1], every standard system of parameters forms a d -sequence, and Proposition 3.8 is known by Kawasaki [K, Lemma 2.2] for standard systems of parameters. We include a brief proof for the reader's convenience.

Proposition 3.8. *Suppose M has finite local cohomologies and let \mathfrak{a} be a standard ideal for M . Let $x_1, \dots, x_r, \dots, x_{r+n}$ ($0 \leq r < d$, $n \geq 0$) be part of a system of parameters for M . If $(x_1, \dots, x_{r+n})A \subseteq \mathfrak{a}$ then*

$$U((x_1, \dots, x_r)M) \cap (x_1, \dots, x_{r+n})M = (x_1, \dots, x_r)M.$$

Proof. We go by induction on n . The result is trivial when $n = 0$.

Suppose $n > 0$. Let a be in $U((x_1, \dots, x_r)M) \cap (x_1, \dots, x_{r+n})M$. Write $a = \sum_{i=1}^{r+n} x_i a_i$ with each a_i in M . Since

$$U((x_1, \dots, x_r)M) = ((x_1, \dots, x_r)M :_M x_{r+n}),$$

we multiply a by x_{r+n} and obtain

$$\sum_{i=1}^{r+n} x_i x_{r+n} a_i \in (x_1, \dots, x_r)M.$$

Examining the highest term, we see that $x_{r+n}^2 a_{r+n}$ is in $(x_1, \dots, x_{r+n-1})M$. Since \mathfrak{a} is a standard ideal for M , this implies that

$$x_{r+n} a_{r+n} \in (x_1, \dots, x_{r+n-1})M.$$

Recalling our expression for a , we see that a is in $(x_1, \dots, x_{r+n-1})M$. Thus a is in $U((x_1, \dots, x_r)M) \cap (x_1, \dots, x_{r+n-1})M$. By the induction hypothesis, we see that a is in $(x_1, \dots, x_r)M$, and the proof is complete. \square

3.4. Local cohomology as a quotient of an unmixed component. We thank the referee for pointing out the following well-known lemma and its proof; the lemma is more general than the one we originally included.

Lemma 3.9. *Suppose M has finite local cohomologies with $t = \text{depth } M$ and let x_1, \dots, x_t be part of a system of parameters for M . Then x_1, \dots, x_t is a regular sequence on M .*

Proof. The submodule $(x_1, \dots, x_{i-1})M$ is unmixed up to \mathfrak{m} -primary components for all integers $1 \leq i \leq t$. Therefore x_i is an $M/(x_1, \dots, x_{i-1})M$ -regular element for all integers $1 \leq i \leq t$. \square

Remark 3.10. Suppose M has finite local cohomologies and let \mathfrak{a} be a standard ideal for M . Let x_1, \dots, x_d be a system of parameters for M and let \mathfrak{q} denote the ideal they generate. According to [SV, Theorem and Definition 17, p. 261], if $\mathfrak{q} \subseteq \mathfrak{a}$, then we have

$$\mathfrak{q} H_{\mathfrak{m}}^i(M/(x_1, \dots, x_j)M) = 0$$

for all i and j with $j \geq 0$ and $0 \leq i < d - j$.

Lemma 3.11. *Suppose x is an element of A that is regular on M . If x annihilates $H_{\mathfrak{m}}^r(M)$ and if $H_{\mathfrak{m}}^{r-1}(M) = 0$, then*

$$H_{\mathfrak{m}}^r(M) \cong H_{\mathfrak{m}}^{r-1}(M/xM).$$

Proof. From the short exact sequence

$$0 \longrightarrow M \longrightarrow M \longrightarrow M/xM \longrightarrow 0$$

induced by multiplication by x on M , we obtain the following isomorphism from the long exact sequence for local cohomology:

$$0 \longrightarrow H_{\mathfrak{m}}^{r-1}(M/xM) \longrightarrow H_{\mathfrak{m}}^r(M) \longrightarrow 0.$$

The zero on the left side comes from the fact that $H_{\mathfrak{m}}^{r-1}(M) = 0$; the zero on the right side is due to the fact that $xH_{\mathfrak{m}}^r(M) = 0$. \square

Lemma 3.12. *If x_1, \dots, x_r is a regular sequence on M such that $(x_1, \dots, x_r)A$ annihilates $H_{\mathfrak{m}}^r(M)$, then*

$$H_{\mathfrak{m}}^r(M) \cong H_{\mathfrak{m}}^0(M/(x_1, \dots, x_r)M).$$

Proof. We go by induction on r . If $r = 1$ then we are done by Lemma 3.11.

Now assume $r > 1$. By Lemma 3.11, we have an isomorphism $H_{\mathfrak{m}}^r(M) \cong H_{\mathfrak{m}}^{r-1}(M/x_1M)$. Set $N = M/x_1M$. By the induction hypothesis applied to N and the sequence x_2, \dots, x_r , we have $H_{\mathfrak{m}}^{r-1}(N) \cong H_{\mathfrak{m}}^0(N/(x_2, \dots, x_r)N)$. We complete the proof using this and the previous isomorphism. \square

Proposition 3.13. *Suppose M has finite local cohomologies and let \mathfrak{a} be a standard ideal for M . Let x_1, \dots, x_r ($0 \leq r \leq d - 1$) be part of a system of parameters for M and suppose $\text{depth } M \geq r$. If $(x_1, \dots, x_r)A \subseteq \mathfrak{a}$ then*

$$H_{\mathfrak{m}}^r(M) \cong U((x_1, \dots, x_r)M) / (x_1, \dots, x_r)M.$$

Proof. By Lemma 3.9, x_1, \dots, x_r forms a regular sequence on M . By Remark 3.10 we have $(x_1, \dots, x_r) H_m^j(M) = 0$ for all j such that $0 \leq j < d$, so from Lemma 3.12 we obtain the isomorphism $H_m^r(M) \cong H_m^0(M/(x_1, \dots, x_r)M)$. We complete the proof with an application of Remark 3.3. \square

3.5. Towards the directness of a sum. This subsection is dedicated to the proof of the following proposition.

Proposition 3.14. *Suppose M has finite local cohomologies and let \mathfrak{a} be a standard ideal for M . Let x_1, \dots, x_d be a system of parameters for M and suppose $(x_1, \dots, x_d)A \subseteq \mathfrak{a}$. If M has positive depth and r is an integer with $1 \leq r \leq \text{depth } M$, then*

$$U((x_1, \dots, x_r)M) \cap \sum_{i=1}^r U((x_1, \dots, \hat{x}_i, \dots, x_d)M) \subseteq (x_1, \dots, x_r)M.$$

Proof. If $r = d$ then M is Cohen-Macaulay. In this case, x_1, \dots, x_r is a system of parameters for M , hence $(x_1, \dots, x_r)M$ is unmixed, so the inclusion is trivial. Now we assume $r < d$.

Let

$$N = U((x_1, \dots, x_r)M) \cap \sum_{i=1}^r U((x_1, \dots, \hat{x}_i, \dots, x_d)M).$$

Since $U((x_1, \dots, x_r)M) = ((x_1, \dots, x_r)M :_M x_d)$ and $U((x_1, \dots, \hat{x}_i, \dots, x_d)M) = ((x_1, \dots, \hat{x}_i, \dots, x_d)M :_M x_i)$, we see that the submodule $x_1 \cdots x_r x_d N$ is contained in

$$x_1 \cdots x_r (x_1, \dots, x_r)M \cap \sum_{i=1}^r x_1 \cdots \hat{x}_i \cdots x_r x_d (x_1, \dots, \hat{x}_i, \dots, x_d)M.$$

By considering $x_d M$ separately, we see that the previous expression equals

$$x_1 \cdots x_r (x_1, \dots, x_r)M \cap \left[\sum_{i=1}^r x_1 \cdots \hat{x}_i \cdots x_r x_d (x_1, \dots, \hat{x}_i, \dots, x_{d-1})M + \sum_{i=1}^r x_1 \cdots \hat{x}_i \cdots x_r x_d^2 M \right].$$

We enlarge the product on the left and see that the last expression is contained in

$$(3.1) \quad \begin{aligned} & (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-1})M \\ & \cap \left[x_d \sum_{i=1}^r x_1 \cdots \hat{x}_i \cdots x_r (x_1, \dots, \hat{x}_i, \dots, x_{d-1})M \right. \\ & \quad \left. + x_d^2 \sum_{i=1}^r x_1 \cdots \hat{x}_i \cdots x_r M \right]. \end{aligned}$$

Note that

$$\sum_{i=1}^r x_1 \cdots \hat{x}_i \cdots x_r (x_1, \dots, \hat{x}_i, \dots, x_{d-1})M \subseteq (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-1})M,$$

so the first summand in the right side of the intersection (3.1) is contained in the left side of the intersection and so expression (3.1) equals

$$(3.2) \quad x_d \sum_{i=1}^r x_1 \cdots \widehat{x}_i \cdots x_r (x_1, \dots, \widehat{x}_i, \dots, x_{d-1}) M \\ + (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-1}) M \cap x_d^2 \sum_{i=1}^r x_1 \cdots \widehat{x}_i \cdots x_r M.$$

Since \mathfrak{a} is a standard ideal, anything multiplying x_d^2 into the submodule

$$(x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-1}) M$$

also multiplies x_d into this submodule and so the second summand of (3.2) is contained in $x_d (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-1}) M$. (We will use this sort of maneuver again later in the proof, when we are dealing with x_k^2 instead of x_d^2 .) Thus expression (3.2), and hence $x_1 \cdots x_r x_d N$, is contained in

$$x_d (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-1}) M.$$

The depth of M is positive and x_d is part of a system of parameters for M , so by Lemma 3.9, x_d is regular on M . Hence

$$x_1 \cdots x_r N \subseteq (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-1}) M.$$

Now we use a similar technique to show inductively that $x_1 \cdots x_r N$ is contained in the submodules

$$(x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-3}, x_{d-2}) M, \\ (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{d-3}) M, \dots, \\ (x_1^2, \dots, x_r^2, x_{r+1}) M, \text{ and finally} \\ (x_1^2, \dots, x_r^2) M.$$

Suppose we have

$$(3.3) \quad x_1 \cdots x_r N \subseteq (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_k) M$$

where $r+2 \leq k \leq d-1$. Our goal is to see that $x_1 \cdots x_r N$ is contained in $(x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M$. Since $U((x_1, \dots, x_r) M) = ((x_1, \dots, x_r) M :_M x_k)$, the submodule $x_1 \cdots x_r x_k N$ is contained in $x_1 \cdots x_r (x_1, \dots, x_r) M$. According to Equation (3.3), we see that $x_1 \cdots x_r x_k N$ is contained in

$$x_k (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M + x_k^2 M.$$

Hence $x_1 \cdots x_r x_k N$ is contained in

$$x_1 \cdots x_r (x_1, \dots, x_r) M \cap [x_k (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M + x_k^2 M]$$

Enlarging the left side of the intersection, we obtain

$$(x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M \\ \cap [x_k (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M + x_k^2 M]$$

Since $x_k (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M \subseteq (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M$, the previous expression becomes

$$x_k (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M \\ + (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M \cap x_k^2 M.$$

Since \mathfrak{a} is a standard ideal, this last expression is just

$$x_k (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M$$

and we have

$$x_1 \cdots x_r x_k N \subseteq x_k (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M$$

Since x_k is regular on M by Lemma (3.9), we now see that

$$x_1 \cdots x_r N \subseteq (x_1^2, \dots, x_r^2, x_{r+1}, \dots, x_{k-1}) M.$$

This completes the inductive portion of the proof, and we now see that

$$x_1 \cdots x_r N \subseteq (x_1^2, \dots, x_r^2, x_{r+1}) M.$$

Using the previous inclusion and the same type of argument applied to the submodule $x_1 \cdots x_r x_{r+1} N$, we may obtain $x_1 \cdots x_r N \subseteq (x_1^2, \dots, x_r^2) M$. Since M has depth at least r and x_1, \dots, x_r is part of a system of parameters for M , x_1, \dots, x_r is actually a regular sequence on M (in any order). A simple proof using this fact shows that $N \subseteq (x_1, \dots, x_r) M$, as desired. This completes the proof. \square

4. EXAMPLES

The first subsection in this section gives examples of rings to which our main theorem applies. The second subsection gives an example to show that eventual constant index of reducibility does not imply finite local cohomologies in general.

4.1. Some Rings Having Finite Local Cohomologies. It is easy to produce examples for the Main Theorem of this paper in the case $d = 1$: Any Noetherian local ring of dimension one will do. More care is required in case $d > 1$; the proposition we quote below provides us with plenty of examples.

Proposition 4.1. [EG, Theorem A]

Let k be an infinite field and let $R = k[X_1, \dots, X_n]$ be the full ring of polynomials over k with n at least 4. Suppose that $2 \leq t_1 < \dots < t_s \leq n - 2$ is a sequence of integers and that L_1, \dots, L_s are graded R -modules of finite length. Then there is a graded prime ideal I such that $H_m^i(R/I)$ is zero unless $i = t_1 - 1, \dots, t_s - 1, n - 2$, while $H_m^{t_j - 1}(R/I)$ is isomorphic to L_j for $j = 1, \dots, s$. Moreover, if t_1 is at least 3, then R/I may also be taken to be a normal domain.

In order to obtain examples relevant to this paper, we take $s = 1$ and $t_1 = r + 1$, where r is the integer mentioned in the Main Theorem. We may then choose L_1 to be any R -module of finite length; L_1 will be the r th local cohomology module of R/I and all other local cohomology modules of R/I will be zero, except of course for the highest, at position $n - 2$. To obtain local examples, we can localize at the maximal homogeneous ideal $(X_1, \dots, X_n) R$.

Now we turn to a more concrete example which is suggested to us by Goto. Let k be a field and let s and t be indeterminates. In what follows, we work with a graded ring over a field instead of a local ring; to obtain a local example, we can localize at the maximal homogeneous ideal. Set $A = k[s^5, s^4t, st^4, t^5]$ and let \mathfrak{m} denote the maximal homogeneous ideal. Then A is a two-dimensional graded domain, where the grading is done by total degree. We will show that A has finite local cohomologies, that \mathfrak{m} does not kill the first local cohomology module of A , and that A does not have constant index of reducibility of parameter ideals.

To see that A has finite local cohomologies, according to Proposition 2.1, it suffices to produce a standard system of parameters. Let $x = s^{10}$ and $y = t^{10}$; we claim that $\{x, y\}$ is a standard system of parameters. Let $\mathfrak{q} = (x, y)A$. According to [SV, Theorem and Definition 17, p. 261], it suffices to show that each of the four sequences $\{x, y\}$, $\{x^2, y\}$, $\{x, y^2\}$, and $\{x^2, y^2\}$ is \mathfrak{q} -weak. Since x and y are regular elements, we only need to check $(xA :_A y) = (xA :_A \mathfrak{q})$, $(x^2A :_A y) = (x^2A :_A \mathfrak{q})$, $(xA :_A y^2) = (xA :_A \mathfrak{q})$, and $(x^2A :_A y^2) = (x^2A :_A \mathfrak{q})$. Note that these ideals are monomial ideals, so they can be computed directly. In particular, we have

$$\begin{aligned} (xA :_A y^2) &= (s^{10}A :_A t^{20}) = s^{10}A + (s^{13}t^2, s^{12}t^3)A \\ (x^2A :_A y^2) &= (s^{20}A :_A t^{20}) = s^{20}A + (s^{23}t^2, s^{22}t^3)A \end{aligned}$$

It is not difficult to see $s^{13}t^2 \cdot t^{10} = s^{13}t^{12} = s^{10} \cdot s^3t^{12}$ and $s^{12}t^3 \cdot t^{10} = s^{12}t^{13} = s^{10} \cdot s^2t^{13}$ are contained in $xA = s^{10}A$, so we have $s^{13}t^2, s^{12}t^3 \in (xA :_A y)$. Thus $(xA :_A y^2) = (xA :_A y) = (xA :_A \mathfrak{q})$. Similarly, we see that $s^{23}t^2$ and $s^{22}t^3$ are in $(x^2A :_A y)$, so $(x^2A :_A y^2) = (x^2A :_A y)$. Moreover, we also see that $s^{23}t^2$ and $s^{22}t^3$ are in $(x^2A :_A x)$. Thus we have $(x^2A :_A y^2) = (x^2A :_A y) = (xA :_A \mathfrak{q})$.

Since $\{x, y\}$ is a standard system of parameters, by [SV, Theorem and Definition 17, p. 261], they each kill $H_m^1(A)$. Moreover, since the ideal xA is unmixed up to \mathfrak{m} -primary components, we have $H_m^1(A) \cong U(xA)/xA$. On the other hand, by Lemma 3.4, we see that $U(xA) = (xA :_A y) = (s^{10}, s^{12}t^3, s^{13}t^2)A$; thus $H_m^1(A) \cong (s^{10}, s^{12}t^3, s^{13}t^2)A/s^{10}A$. From this we see that $H_m^1(A)$ is not killed by \mathfrak{m} but it is killed by \mathfrak{m}^2 . Furthermore, we compute the socle of the first local cohomology by calculating $(s^{10} :_A \mathfrak{m}) = (s^{10}, s^{13}t^7, s^{17}t^3)A$. Hence the socle dimension of $H_m^1(A)$ is two.

Now we use the Čech complex to see that the socle dimension of $H_m^2(A)$ is four. According to the Čech complex, we have

$$H_m^2(A) \cong \frac{A[1/s^5t^5]}{A[1/s^5] + A[1/t^5]}.$$

We examine the components of this expression and find that

$$\begin{aligned} A[1/s^5] &= k[t/s] [s^5, 1/s^5] = \sum_{n \in \mathbb{Z}} k[t/s]s^{5n}, \\ A[1/t^5] &= k[s/t] [t^5, 1/t^5] = \sum_{n \in \mathbb{Z}} k[s/t]t^{5n}, \end{aligned}$$

and

$$A[1/s^5t^5] = \sum_{n \in \mathbb{Z}} \sum_{\substack{\alpha + \beta = 5n \\ \alpha, \beta \in \mathbb{Z}}} k s^\alpha t^\beta.$$

Notice that $1/st^4$, $1/s^2t^3$, $1/s^3t^2$, and $1/s^4t$ are linearly independent and contained in the socle. Now we just need to show that nothing outside of their span is in the socle. To see this, suppose $\alpha + \beta \geq 10$ and $\alpha, \beta \geq 1$. We show that $1/s^\alpha t^\beta$ is not in the socle. It must be true that either $\alpha - 4 > 0$ or $\beta - 4 > 0$. By symmetry, we may assume without loss of generality that $\alpha - 4 > 0$. If $\beta = 1$, then $\alpha \geq 9$, and the element $s^5(1/s^\alpha t^\beta) = 1/s^{\alpha-5}t$ is not in $A[1/s^5] + A[1/t^5]$, since its denominator is not a pure power of s or t . Thus we see that when $\beta = 1$, the element $1/s^\alpha t^\beta$ is not in the socle. If $\beta > 1$, then $s^4t(1/s^\alpha t^\beta) = 1/s^{\alpha-4}t^{\beta-1}$ is not in $A[1/s^5] + A[1/t^5]$, and we see that $1/s^\alpha t^\beta$ is not in the socle. Hence $\text{Socdim}(H_m^2(A)) = 4$.

According to the Main Theorem, the index of reducibility of parameter ideals for A contained in high powers of the maximal ideal is $2 \times 2 + 4 = 8$. Using the fact that the index of reducibility of a parameter ideal \mathfrak{q} equals the socle dimension of A/\mathfrak{q} , we find that $N_A((s^5, t^5)A; A) = 2$, $N_A((s^{10}, t^{10})A; A) = 4$, and $N_A((s^{15}, t^{15})A; A) = 8$, thus A does not have constant index of reducibility. More precisely, with direct computation, we have

$$\begin{aligned} ((s^5, t^5)A :_A \mathfrak{m}) &= (s^5, t^5, s^{12}t^3, s^3t^{12})A \\ ((s^{10}, t^{10})A :_A \mathfrak{m}) &= (s^{10}, t^{10}, s^{17}t^3, s^{13}t^7, s^7t^{13}, s^3t^{17})A \\ ((s^{15}, t^{15})A :_A \mathfrak{m}) &= (s^{15}, t^{15}, s^{22}t^3, s^{18}t^7, s^{14}t^{11}, s^{13}t^{12}, \\ &\quad s^{12}t^{13}, s^{11}t^{14}, s^7t^{18}, s^3t^{22})A, \end{aligned}$$

so

$$\begin{aligned} N_A((s^5, t^5)A; A) &= \text{Socdim}(A/(s^5, t^5)A) = 2 \\ N_A((s^{10}, t^{10})A; A) &= \text{Socdim}(A/(s^{10}, t^{10})A) = 4 \\ N_A((s^{15}, t^{15})A; A) &= \text{Socdim}(A/(s^{15}, t^{15})A) = 8 \end{aligned}$$

We can also confirm certain assertions in this example by using the computer program Macaulay 2 [M].

Remark 4.2. The preceding example shows that parameter ideals contained in a standard ideal need not have the same index of reducibility. Indeed, $(s^{10}, t^{10})A$ and $(s^{15}, t^{15})A$ are both contained in the standard ideal $(s^{10}, t^{10})A$, but they have different indexes of reducibility.

4.2. Eventual Constancy Does Not Imply Finite Local Cohomologies.

Eventual constant index of reducibility does not imply finite local cohomologies. In fact, even constant index of reducibility does not imply finite local cohomologies, as one can see from [GSu, Example 4.7]. In this section we provide another example by slightly generalizing and reexamining an example from Section 5 of [GSaI].

Example 4.3. Let $n \geq 2$ and let S be an $n + 1$ -dimensional regular local ring whose maximal ideal is $(x, y_1, y_2, \dots, y_n)S$. Let $A = S/(xy_1, xy_2, \dots, xy_n)S$ and let $\mathfrak{m} = (x, y_1, y_2, \dots, y_n)A$, the maximal ideal of A . Then A is a Noetherian local ring with dimension n and depth 1 such that every parameter ideal contained in \mathfrak{m}^2 has index of reducibility 2, and A does not have finite local cohomologies.

We note that this example does not have constant index of reducibility, since the parameter ideal $(y_1 - x, y_2, \dots, y_n)A$ is irreducible.

Proof. Let $\mathfrak{p} = (y_1, y_2, \dots, y_n)A$. The ring A does not have finite local cohomologies since its minimal primes xA and \mathfrak{p} have different dimensions: $\dim A/(x) = n$ and $\dim A/\mathfrak{p} = 1$ [SV, Proposition 16, p. 260].

Let \mathfrak{q} be a parameter ideal for A contained in \mathfrak{m}^2 . Consider the following exact sequence of A -modules:

$$0 \longrightarrow A/\mathfrak{p} \xrightarrow{-x} A \longrightarrow A/xA \longrightarrow 0.$$

Since $A/xA \cong S/xS$, A/xA is a Cohen-Macaulay A -module of dimension n . Since \mathfrak{q} is still a parameter ideal for A/xA , it is generated by an A/xA -regular sequence,

so $\mathrm{Tor}_A^1(A/\mathfrak{q}, A/xA) = 0$. We tensor with A/\mathfrak{q} and use fundamental isomorphisms to obtain an exact sequence

$$0 \longrightarrow A/(\mathfrak{p} + \mathfrak{q}) \xrightarrow{\cdot x} A/\mathfrak{q} \longrightarrow A/(xA + \mathfrak{q}) \longrightarrow 0.$$

Now we apply the socle functor:

$$0 \longrightarrow \mathrm{Soc}(A/(\mathfrak{p} + \mathfrak{q})) \xrightarrow{\cdot x} \mathrm{Soc}(A/\mathfrak{q}) \longrightarrow \mathrm{Soc}(A/(xA + \mathfrak{q})).$$

Since A/\mathfrak{p} is a DVR and $A/(\mathfrak{p} + \mathfrak{q})$ has finite length, it follows that

$$\mathrm{Socdim}(A/(\mathfrak{p} + \mathfrak{q})) = 1.$$

Since A/xA is a type 1 Cohen-Macaulay A -module and \mathfrak{q} is a parameter ideal for A/xA , $\mathrm{Socdim}(A/(xA + \mathfrak{q})) = 1$. Thus, as soon as we show the map $A/\mathfrak{q} \longrightarrow A/(xA + \mathfrak{q})$ is surjective on the socles, we will have $N_A(\mathfrak{q}; A) = 1 + 1 = 2$, and our proof will be complete.

Write $\mathfrak{q} = (f_1, f_2, \dots, f_n)A$. We will modify these elements f_i during the course of the proof. Use $\bar{\cdot}$ to denote reduction modulo \mathfrak{p} . Since \bar{A} is a DVR, we know that one of the ideals $\bar{f}_i \bar{A}$ contains the others. Without loss of generality, we assume it is $\bar{f}_1 \bar{A}$. Thus there are elements $u_i \in A$ such that $\bar{f}_i = \bar{u}_i \bar{f}_1$ for $2 \leq i \leq n$. Hence $f_i - u_i f_1 \in \mathfrak{p}$ for each $2 \leq i \leq n$. Since $\mathfrak{q} = (f_1, f_2 - u_2 f_1, \dots, f_n - u_n f_1)A$, we may replace f_i by $f_i - u_i f_1$ for $2 \leq i \leq n$ to arrive at a situation where $\mathfrak{q} = (f_1, f_2, \dots, f_n)A$ with each of f_2, \dots, f_n in \mathfrak{p} .

Since \mathfrak{q} is not contained in \mathfrak{p} , we know that f_1 is not in \mathfrak{p} . Since $\mathfrak{m} = xA + \mathfrak{p}$, we may write $f_1 = \epsilon x^m + g$ with ϵ a unit in A and $g \in \mathfrak{p}$. Thus $\mathfrak{q} = (x^m + \epsilon^{-1}g, f_2, \dots, f_n)A$. We replace f_1 by $\epsilon^{-1}g$ to arrive at a situation where $\mathfrak{q} = (x^m + f_1, f_2, \dots, f_n)A$ with each f_i in \mathfrak{p} . Furthermore, since $\mathfrak{q} \subseteq \mathfrak{m}^2$, we know that $m \geq 2$ and each f_i is in \mathfrak{m}^2 .

Let a be an element of A that maps to a socle generator for $A/(xA + \mathfrak{q})$; thus $a\mathfrak{m} \subseteq xA + \mathfrak{q}$. Our goal is to show that $a\mathfrak{m} \subseteq \mathfrak{q}$. Since the image of \mathfrak{m} and \mathfrak{p} are the same in A/xA , we may assume $a \in \mathfrak{p}$. Since $xA \cap \mathfrak{p} = 0$, $ax = 0$. It remains to see that $ay_i \in \mathfrak{q}$ for each $1 \leq i \leq n$.

Since $a\mathfrak{m} \subseteq \mathfrak{q} + xA = (f_1, f_2, \dots, f_n, x)A$, we have equations

$$ay_i = u_{i1}f_1 + u_{i2}f_2 + \dots + u_{in}f_n + v_i x$$

for $1 \leq i \leq n$ and elements u_{ij}, v_i in A . We see that each element $v_i x$ is in $xA \cap \mathfrak{p}$, and is thus zero.

We may assume that each element u_{ij} is either a unit or is actually in \mathfrak{p} . First we show that it is not possible for one of these first coefficients to be a unit; we go by way of contradiction. Suppose one of the first coefficients u_{i1} is a unit; without loss of generality, suppose it is u_{11} . Then we may solve for f_1 and write

$$f_1 = u_{11}^{-1}ay_1 - u_{11}^{-1}u_{12}f_2 - \dots - u_{11}^{-1}u_{1n}f_n.$$

Hence

$$\mathfrak{q} = (x^m + u_{11}^{-1}ay_1, f_2, \dots, f_n).$$

Since $u_{11}^{-1}a$ maps to a socle element of $A/(xA + \mathfrak{q})$ as well, we may replace a by $u_{11}^{-1}a$. Now we have $\mathfrak{q} = (x^m + ay_1, f_2, \dots, f_n)$.

Set $R = A/(x, f_2, \dots, f_n)A$. Then R is a one-dimensional Noetherian local ring with maximal ideal $\mathfrak{n} = \mathfrak{m}R$ and a parameter ideal $Q = ay_1R$ such that the socle of R/Q contains the image of a . Furthermore, since R is the quotient of the regular local ring A/xA by elements contained in the square of the maximal ideal, the

multiplicity of R is greater than 1. Set $I = (Q :_R \mathfrak{n})$. Then according to [GSaI, Proposition 2.3], we have that $\mathfrak{n}I = \mathfrak{n}Q$. Since $a \in I$, $\mathfrak{n}a \subseteq \mathfrak{n}I = \mathfrak{n}Q = \mathfrak{n}ay_1$. Thus $\mathfrak{n}a = \mathfrak{n}ay_1$, so by Nakayama's Lemma we see that $\mathfrak{n}a = 0$. This is a contradiction, since $Q = ay_1R \subseteq \mathfrak{n}a$, and Q is a parameter ideal in the one-dimensional ring R . Thus we see that none of the first coefficients u_{i1} can be a unit.

Now, since each of the n first coefficients u_{i1} , $1 \leq i \leq n$, is a nonunit, we may assume they are all in \mathfrak{p} , and we may rewrite the equations as

$$ay_i = u_{i1}(x^m + f_1) + u_{i2}f_2 + \cdots + u_{in}f_n.$$

Thus we see that $\mathfrak{a}\mathfrak{m} \subseteq \mathfrak{q}$, as desired. \square

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