

# THE INDEX OF REDUCIBILITY OF PARAMETER IDEALS IN LOW DIMENSION

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ABSTRACT. In this paper we present results concerning the following question: If  $M$  is a finitely-generated module with finite local cohomologies over a Noetherian local ring  $(A, \mathfrak{m})$ , does there exist an integer  $\ell$  such that every parameter ideal for  $M$  contained in  $\mathfrak{m}^\ell$  has the same index of reducibility? We show that the answer is *yes* if  $\dim M = 1$  or if  $\dim M = 2$  and  $\text{depth } M > 0$ . This research is closely related to work of Goto-Suzuki and Goto-Sakurai; Goto-Sakurai have supplied an answer of *yes* in case  $M$  is Buchsbaum.

## 1. INTRODUCTION

Let  $A$  be a  $d$ -dimensional Noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ , and let  $M$  be a finitely generated  $A$ -module. Recall that a submodule of  $M$  is called *irreducible* if it cannot be written as the intersection of two larger submodules. It is well known that every submodule  $N$  of  $M$  can be expressed as an irredundant intersection of irreducible submodules, and that the number of irreducible submodules appearing in such an expression depends only on  $N$  and not on the expression [ShV, p. 92-3].

For an ideal  $I$  of  $A$ , we say that  $I$  is *cofinite* on  $M$  if the module  $M/IM$  has finite length. For an ideal  $I$  of  $A$  which is cofinite on  $M$ , the *index of reducibility* of  $I$  on  $M$  is defined as the number of submodules appearing in an irredundant expression of  $IM$  as an intersection of irreducible submodules of  $M$ . We denote the index of reducibility of  $I$  on  $M$  by  $r_A(I; M)$ .

The smallest number of generators of an ideal which is cofinite on  $M$  is the dimension of  $M$ , and a cofinite ideal having this minimal number of generators is called a *parameter ideal* for  $M$ . Our interest in the index of reducibility of parameter ideals stems from the relationship with the Cohen-Macaulay and Gorenstein properties. In 1956, D. G. Northcott proved that in a Cohen-Macaulay local ring, the index of reducibility of any parameter ideal depends only on the ring [N, Theorem 3]. This result extends to modules, and the common index of reducibility of parameter ideals for a Cohen-Macaulay module  $M$  is called the (*Cohen-Macaulay type*) of  $M$ . We denote the type of  $M$  by  $r_A(M)$ .

As a partial converse, Northcott along with D. Rees proved in [NR1] that if every parameter ideal of  $A$  is irreducible, then  $A$  is Cohen-Macaulay. This provides an attractive characterization of a Gorenstein local ring as a local ring in which every parameter ideal is irreducible.

One might suspect that the Cohen-Macaulay property of a local ring is characterized by the constant index of reducibility of parameter ideals. However, in 1964,

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S. Endo and M. Narita in [EN] gave an example of a Noetherian local ring in which the index of reducibility of each parameter ideal is two, and yet the ring is not Cohen-Macaulay.

In 1984, S. Goto and N. Suzuki generalized the example of Endo-Narita, as well as undertaking a study of the supremum of the index of reducibility of parameter ideals for  $M$  [GSu]. We refer to this supremum as the *Goto-Suzuki type* (*GS-type*) of  $M$ , and denote it by  $\bar{r}_A(M)$ . In the case where  $M$  is Cohen-Macaulay, the GS-type  $\bar{r}_A(M)$  is equal to the type  $r_A(M)$ . However, Goto-Suzuki provide examples where the GS-type of a Noetherian local ring is infinity.

We introduce some terminology in order to state one of the main results of Goto-Suzuki. We denote the  $i$ th local cohomology module of  $M$  with respect to  $\mathfrak{m}$  by  $H_{\mathfrak{m}}^i(M)$ . We say that  $M$  has *finite local cohomologies* if the modules  $H_{\mathfrak{m}}^i(M)$  have finite length for  $i \neq d$ . We use  $\lambda_A(M)$  to denote the length of  $M$ , and we set  $\mathfrak{S}(M) = \lambda_A(\text{Hom}_A(k, M))$ , the *socle dimension* of  $M$ . We let  $\mu_A(M)$  denote the minimal number of generators of  $M$ , and let  $E$  denote the injective hull of the residue field  $k$ . The main result of Goto-Suzuki concerning the GS-type of a module having finite local cohomologies is the following:

**Theorem 1.1** (Goto-Suzuki). *Let  $M$  be a finitely generated  $d$ -dimensional  $A$ -module with finite local cohomologies. Then we have the following inequalities:*

$$(1.1) \quad \sum_{i=0}^d \binom{d}{i} \mathfrak{S}(H_{\mathfrak{m}}^i(M)) \leq \bar{r}_A(M) \leq \sum_{i=0}^{d-1} \binom{d}{i} \lambda_A(H_{\mathfrak{m}}^i(M)) + \mu_{\hat{A}}(K),$$

where  $\hat{A}$  is the  $\mathfrak{m}$ -adic completion of  $A$  and  $K = \text{Hom}_A(H_{\mathfrak{m}}^d(M), E)$  is the canonical module of the completion of  $M$ .

*Proof.* See [GSu, Theorems 2.1 and 2.3]. □

In 1994, Kawasaki used this result to determine conditions under which a module having finite GS-type is Cohen-Macaulay. The main result of Kawasaki in [K] is that if  $A$  is the homomorphic image of a Cohen-Macaulay ring, then  $M$  is Cohen-Macaulay if and only if  $\bar{r}_A(M)$  is finite,  $M_{\mathfrak{p}}$  is Cohen-Macaulay for all primes in the support of  $M$  with  $\dim M_{\mathfrak{p}} < \bar{r}_A(M)$ , and all the associated primes of  $M$  have the same dimension.

In two recent papers [GS1, GS2], Goto with H. Sakurai has returned to the study of the index of reducibility of parameter ideals in order to investigate when the equality  $I^2 = QI$  holds for a parameter ideal  $Q$  in  $A$ , where  $I = (Q :_A \mathfrak{m})$ . According to earlier research of A. Corso, C. Huneke, C. Polini, and W. Vasconcelos [CHP, CP, CPV], this equality holds for all parameter ideals  $Q$  in case  $A$  is a Cohen-Macaulay ring which is not regular. Goto-Sakurai generalize this and say that if  $A$  is a Buchsbaum ring whose multiplicity is greater than 1, then the equality  $I^2 = QI$  holds for any parameter ideal whose index of reducibility is the GS-type of  $A$ . Thus, if  $A$  is a Buchsbaum ring whose multiplicity is greater than 1 and if  $A$  has constant index of reducibility of parameter ideals, then the equality  $I^2 = QI$  holds for all parameter ideals  $Q$ .

Most pertinent to our discussion is Corollary 3.13 of Goto-Sakurai in [GS1], which states that if  $A$  is a Buchsbaum ring of positive dimension, then there is an integer  $\ell$  such that the index of reducibility of  $Q$  is independent of  $Q$  and equals

$\bar{r}_A(A)$  for all parameter ideals  $Q \subseteq \mathfrak{m}^\ell$ . In view of this, it is natural to ask the following question:

**Question 1.2.** *Suppose  $(A, \mathfrak{m})$  is a Noetherian local ring having finite local cohomologies. Is there an integer  $\ell$  such that the index of reducibility of any parameter ideal contained in  $\mathfrak{m}^\ell$  is the same?*

We note that if  $A$  is a ring satisfying the hypothesis of the question, and if the answer to the question is *yes* for  $A$ , then the common index of reducibility of parameter ideals in high powers of the maximal ideal is equal to the lower bound of Goto-Suzuki:  $\sum_{i=0}^d \binom{d}{i} \mathfrak{S}(\mathbb{H}_{\mathfrak{m}}^i(A))$ . This is because implicit in the proof of the lower bound [GSu, Theorem 2.3] we find that given any system of parameters  $x_1, \dots, x_d$  for  $A$ , there are integers  $n_i$  such that the parameter ideal  $(x_1^{n_1}, \dots, x_d^{n_d})A$  has index of reducibility  $\sum_{i=0}^d \binom{d}{i} \mathfrak{S}(\mathbb{H}_{\mathfrak{m}}^i(A))$ .

The main result of the current paper is the following:

**Theorem 1.3.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring and let  $M$  be a finitely-generated  $A$ -module of dimension  $d \leq 2$ . Suppose either  $M$  has dimension 1, or  $M$  has finite local cohomologies and depth at least one. Then there exists an integer  $\ell$  such that for every parameter ideal  $\mathfrak{q}$  for  $M$  contained in  $\mathfrak{m}^\ell$ , the index of reducibility of  $\mathfrak{q}$  on  $M$  is independent of  $\mathfrak{q}$  and is given by*

$$(1.2) \quad N_A(\mathfrak{q}; M) = \sum_{i=0}^d \binom{d}{i} \mathfrak{S}(\mathbb{H}_{\mathfrak{m}}^i(M)).$$

*Proof.* See Theorem 2.3 and 3.3. □

As a corollary of this result we prove that a Noetherian local ring  $A$  of dimension at most 2 is Gorenstein if and only if  $A$  has finite local cohomologies, and inside every power of the maximal ideal of  $A$  there exists an irreducible parameter ideal. A Noetherian local ring having the property that every power of its maximal ideal contains an irreducible parameter ideal is called an *approximately Gorenstein* ring, or is said to have *small cofinite irreducibles (SCI)*. The author's original motivation for studying the index of reducibility of parameter ideals comes from questions that arose while studying M. Hochster's paper [H] exposing the relationship between modules having SCI and the condition that cyclic purity implies purity.

We present an example of a complete Noetherian local ring of dimension  $d$  and depth  $d - 1$  ( $d > 1$ ), such that  $\mathbb{H}_{\mathfrak{m}}^{d-1}(A)$  is not finitely generated, and such that in every power of the maximal ideal there is a parameter ideal with index of reducibility 2 and a parameter ideal with index of reducibility 3. This example is obtained as an idealization, so we present several basic results relating the index of reducibility with an idealization. Using a result of C. Lech [L], we are able to obtain such an example among Noetherian local domains.

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## 2. BACKGROUND AND DIMENSION ONE

We begin with some terminology.

**Definitions and Notation 2.1.** Let  $A$  denote a Noetherian local ring with maximal ideal  $\mathfrak{m}$  and residue field  $k = A/\mathfrak{m}$ , let  $I$  denote an ideal of  $A$ , and let  $M$  denote a finitely generated  $A$ -module.

- (1) The *socle* of  $M$  is defined to be  $S(M) = (0 :_M \mathfrak{m})$ . Note that  $S(M) \cong \text{Hom}_A(k, M)$ . The socle is naturally a vector space over  $k$ , and we denote its dimension by  $\mathfrak{S}(M)$ .
- (2) Suppose  $M$  has depth  $t$ . The *type* of  $M$  is defined by

$$r_A(M) = \dim_k \text{Ext}_A^t(k, M).$$

Note that if  $I$  is cofinite on  $M$ , then  $N_A(I; M) = r_A(M/IM)$ , and if  $M$  has depth 0, then  $\mathfrak{S}(M) = r_A(M)$ . See [BH, p. 13] for more information on the type of a module.

- (3) Given an  $A$ -algebra  $B$ , we define  $M_B = M \otimes_A B$ . Then  $(A/I)_B \cong B/IB$ , and if  $B$  is flat over  $A$ , then for any submodule  $N$  of  $M$ , we have  $M_B/N_B \cong (M/N)_B$ .
- (4) We say that an ideal  $J$  of  $A$  is a *reduction* of  $I$  if  $J \subseteq I$  and there is some integer  $n$  such that  $I^{n+1} = JI^n$ . We say that  $J$  is a *minimal reduction* of  $I$  if there is no reduction of  $I$  properly contained in  $J$ . The *reduction number* of  $I$  with respect to  $J$  is defined as

$$\text{red}_J(I) = \min\{n : I^{n+1} = JI^n\}.$$

We define the *reduction number* of  $I$  to be

$$\text{red}(I) = \min\{\text{red}_J(I) : J \text{ is a minimal reduction of } I\}.$$

We must reduce to the case that the residue field is infinite in order to obtain a principal reduction of the maximal ideal. Thus our first result is a basic lemma concerning the behavior of the index of reducibility under a flat, local change of base. We omit the proof of Lemma 2.2, but cite [Na, Theorem 19.1], [BH, Proposition 1.2.16] and [M, Theorem 7.4] for relevant related results.

**Lemma 2.2.** *Suppose  $A$  and  $B$  are Noetherian local rings with maximal ideals  $\mathfrak{m}$  and  $\mathfrak{n}$ , respectively, let  $M$  be a finitely generated  $A$ -module, and suppose  $B$  is a flat  $A$ -algebra for which  $\mathfrak{n} = \mathfrak{m}B$ . Let  $I$  be an ideal of  $A$  and let  $N$  be a submodule of  $M$ .*

- (1) *If  $M$  has finite length, then  $M_B$  has finite length, and  $\lambda_B(M_B) = \lambda_A(M)$ .*
- (2) *If  $I$  is an ideal of  $A$  which is cofinite on  $M$ , then  $IB$  is cofinite on  $M_B$ .*
- (3) *If  $I$  is an ideal of  $A$  which is cofinite on  $M$ , then*

$$N_B(IB; M_B) = N_A(I; M).$$

- (4)  *$(N :_M I)_B \cong (N_B :_{M_B} IB)$ .*
- (5)  *$H_n^0(M_B) = H_n^0(M)_B$ .*

Now we are prepared to prove that parameter ideals in high powers of a one dimensional Noetherian local ring all have the same index of reducibility.

**Theorem 2.3.** *Suppose  $A$  is a Noetherian local ring with maximal ideal  $\mathfrak{m}$ , and let  $M$  be a finitely generated  $A$ -module of dimension 1. Set  $W = H_{\mathfrak{m}}^0(M)$ . There is an integer  $\ell$  such that the index of reducibility of any parameter ideal  $\mathfrak{q} \subseteq \mathfrak{m}^\ell$  is independent of  $\mathfrak{q}$  and is given by*

$$N_A(\mathfrak{q}; M) = \mathfrak{S}(M) + r_A(M/W).$$

When  $M = A$  and  $k$  is infinite, the integer  $\ell$  may be taken to be

$$\ell = \max\{c, d\} + 1$$

where  $c = \text{red}(\mathfrak{m})$  and  $d$  is the smallest integer with  $\mathfrak{m}^d \cap W = 0$ .

*Proof.* First we show that it suffices to prove the theorem in the case where the residue field  $k$  is infinite. Suppose  $B$  is any local flat  $A$ -algebra whose maximal ideal  $\mathfrak{n}$  is extended from  $A$  and whose residue field is infinite (for instance,  $B = A[x]_{\mathfrak{m}_A[x]}$  for an indeterminate  $x$ ). Then according to Theorem A.11 on p. 415 of [BH], the  $B$ -module  $M_B$  has dimension 1. Therefore we may apply the theorem to  $M_B$ ; let  $\ell$  be the integer guaranteed by the theorem. Set  $V = H_{\mathfrak{n}}^0(M_B)$ ; then according to Part 5 of Lemma 2.2, we have  $V = W_B$ .

Suppose  $\mathfrak{q}$  is a parameter ideal for  $M$  that is contained in  $\mathfrak{m}^\ell$ . Then  $\mathfrak{q}B$  is a parameter ideal for  $M_B$  which is contained in  $\mathfrak{n}^\ell$ , so we have

$$(2.1) \quad N_B(\mathfrak{q}B; M_B) = \mathfrak{S}(M_B) + r_B(M_B/V).$$

According to Part 3 of Lemma 2.2, the left side of equality 2.1 equals  $N_A(\mathfrak{q}; M)$ , so it remains to see that  $\mathfrak{S}(M_B) = \mathfrak{S}(M)$  and  $r_B(M_B/V) = r_A(M/W)$ . For the first equality, if  $W = 0$  then  $V = 0$  and we have nothing to show. Otherwise,  $M$  has depth 0 and thus according to Proposition 1.2.16 part (a) from [BH], so does  $M_B$ , so by part (b) of the same Proposition, we have

$$\mathfrak{S}(M_B) = r_B(M_B) = r_A(M) = \mathfrak{S}(M).$$

To see the equality  $r_B(M_B/V) = r_A(M/W)$ , we first note that

$$M_B/V = M_B/W_B \cong (M/W)_B.$$

Hence, by another application of Proposition 1.2.16 part (b) from [BH], we have

$$r_B(M_B/V) = r_B((M/W)_B) = r_A(M/W).$$

Thus we see that we may assume the residue field  $k$  is infinite.

Our second task is to see that we may replace the ring  $A$  by the ring  $C = A/I$ , where  $I$  is the annihilator of  $M$ . Let  $\mathfrak{q}$  denote an ideal of  $A$  which is a parameter ideal for  $M$ . The  $A$ -module  $M$  becomes an  $C$ -module in a natural way, and the submodule structure of  $M$  remains unchanged. The maximal ideal of  $A$  extends to that of  $C$ , the residue field of  $C$  is  $k$ , and  $\mathfrak{q}C$  is an ideal of  $C$  which is a parameter ideal for  $M$ . If  $N$  is any finitely generated  $A$ -module annihilated by  $I$ , then as sets we have  $\text{Hom}_A(k, N) = \text{Hom}_C(k, N)$ , and we easily check that this equality is actually an isomorphism of  $A$ -modules. Thus  $\mathfrak{S}(N)$  does not depend on whether we view  $N$  as an  $A$ -module or a  $C$ -module. Since the numbers  $N_A(\mathfrak{q}; M)$ ,  $\mathfrak{S}(M)$ , and  $r_A(M/W)$  are all calculated as socle dimensions of quotients of  $M$ , they do not change when we view  $M$  as an  $C$ -module. Thus we replace  $A$  by  $C$  and assume that  $A$  has Krull dimension one.

If  $W = 0$ , then  $M$  is Cohen-Macaulay,  $\mathfrak{S}(M) = 0$ , and for a parameter ideal  $\mathfrak{q} = aA$  we have

$$(2.2) \quad N_A(\mathfrak{q}; M) = \mathfrak{S}(M/aM) = \dim_k \text{Ext}_A^1(k, M) = r_A(M).$$

Thus the proof is complete in this case.

Now suppose  $W \neq 0$ . Since  $W$  has finite length and

$$\bigcap_{i=1}^{\infty} (\mathfrak{m}^i M \cap W) = 0,$$

there is some integer  $d$  with  $\mathfrak{m}^d M \cap W = 0$ .

Since the residue field is infinite,  $\mathfrak{m}$  has a principal reduction; i.e., there is an element  $x \in \mathfrak{m}$  and an integer  $c \geq 1$  such that  $\mathfrak{m}^{c+1} = x\mathfrak{m}^c$  ([BH, Corollary 4.6.10, p. 191]). Set  $\ell = \max\{c, d\} + 1$ . Then we have arrived at a situation where any parameter  $a$  for  $M$  which is in  $\mathfrak{m}^\ell$  is of the form  $a = xy$  with  $y \in \mathfrak{m}^d$ . Furthermore, since we have assumed  $M$  is faithful, the parameters for  $M$  are just the parameters for  $A$  [E, Proposition 10.8, p. 237]. Since the parameters for the one-dimensional ring  $A$  are those elements not in any minimal prime ideal of  $A$ , we see that if  $a = xy$  is a parameter for  $M$ , then so are  $x$  and  $y$ .

Since  $aM \cap W = 0$ , we have  $(W + aM)/aM \cong W$ , so, as in the proof of [GSu, Proposition (2.4)], we see that the top row in the following commutative diagram is exact:

$$(2.3) \quad \begin{array}{ccccccc} 0 & \longrightarrow & W & \longrightarrow & \frac{M}{aM} & \xrightarrow{\beta} & \frac{M}{aM+W} \longrightarrow 0 \\ & & & & \swarrow f & & \uparrow g \\ & & & & & & \frac{M}{xM+W} \end{array}$$

The maps  $f$  and  $g$  are each given by multiplication by  $y$ . The map  $f$  is well-defined since  $y(xM + W) = aM + yW = aM$ , and  $g$  is just the composition  $g = \beta f$ .

To see that  $g$  is injective, view  $g$  as multiplication by  $y$  as follows:

$$\frac{M/W}{x(M/W)} \xrightarrow{g=y} \frac{M/W}{a(M/W)}$$

Since  $M/W$  is a Cohen-Macaulay module and  $y$  is a parameter for  $M$ , and thus for  $M/W$ , we have that  $y$  is regular on  $M/W$ . Hence

$$(xy(M/W) :_{M/W} y) = x(M/W),$$

so that  $g$  is injective.

Information on the dimension of the socles is obtained by applying the functor  $\mathrm{Hom}_A(k, -)$ ; what results is the following exact commutative diagram, where the maps induced by  $f$  and  $g$  are still injective:

$$(2.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(k, W) & \longrightarrow & \mathrm{Hom}_A(k, \frac{M}{aM}) & \xrightarrow{\beta^*} & \mathrm{Hom}_A(k, \frac{M}{aM+W}) \\ & & & & \swarrow \mathrm{Hom}_A(k, f) & & \uparrow \mathrm{Hom}_A(k, g) \\ & & & & & & \mathrm{Hom}_A(k, \frac{M}{xM+W}) \end{array}$$

Here we use  $\beta^*$  to denote  $\mathrm{Hom}_A(k, \beta)$ .

The important point is that since  $x$  and  $a$  are still parameters for the Cohen-Macaulay module  $M/W$ , we have

$$\begin{aligned} \dim_k \mathrm{Hom}_A\left(k, \frac{M}{aM+W}\right) &= \dim_k \mathrm{Hom}_A\left(k, \frac{M/W}{a(M/W)}\right) \\ &= N_A(aA; M/W) \\ &= r_A(M/W), \end{aligned}$$

and similarly

$$\dim_k \operatorname{Hom}_A \left( k, \frac{M}{aM + W} \right) = r_A(M/W).$$

The map  $\operatorname{Hom}_A(k, g)$  is an injection of  $k$ -vector spaces, each of dimension  $r_A(M/W)$ , hence this map is an isomorphism. From the surjectivity of this map, it follows that  $\beta^*$  is surjective, so that the top row of Diagram 2.4 is exact. Since the length of the middle module is  $N_A(aA; M)$  and the left-hand module is the socle of  $W$ , which is the socle of  $M$ , we complete the proof in dimension one using the additivity of length.  $\square$

### 3. DIMENSION 2

We thank the referee for pointing out the fascinating technique in this section, and for pointing out the recent work of Goto-Sakurai [GS1, GS2]. Goto-Sakurai successfully apply this technique to the case of Buchsbaum local rings of arbitrary dimension [GS1, Corollary 3.13].

Suppose  $\mathfrak{q} = (x_1, \dots, x_d)A$  is an ideal generated by a system of parameters  $x_1, \dots, x_d$  for a finitely-generated module  $M$  of dimension  $d$  over a Noetherian local ring  $(A, \mathfrak{m})$ . We may form a direct system of modules by setting  $M_i = M/(x_1^i, \dots, x_d^i)M$  and defining maps from  $M_i \rightarrow M_{i+1}$  given by multiplication by  $x_1 \cdots x_d$ . It is known that the direct limit of this system is  $H_{\mathfrak{m}}^d(M)$  [BH, Theorem 3.5.6]. From this point of view, we see that there is a canonical homomorphism from  $M/\mathfrak{q}M$  to  $H_{\mathfrak{m}}^d(M)$ .

For an ideal  $I$  of a Noetherian local ring  $(A, \mathfrak{m})$  and a finitely-generated module  $M$  we define  $U(I; M)$  to be the unmixed component of the submodule  $IM$  of  $M$ ; that is,  $U(I; M)$  is the intersection of the primary components of the submodule  $IM$  whose associated primes have maximal dimension, equal to  $\dim M/IM$ .

In the course of our proof for dimension 2 we will need to mention several generalizations of the notion of a regular sequence. These definitions can be found in the appendix of [SV], which is a good source for information concerning modules having finite local cohomologies.

**Definition 3.1.** Suppose  $(A, \mathfrak{m})$  is a Noetherian local ring,  $M$  is a finitely-generated  $A$ -module of dimension  $d > 0$ , and let  $\mathfrak{a}$  be an  $\mathfrak{m}$ -primary ideal. A system of elements  $x_1, \dots, x_r$  is called an  $\mathfrak{a}$ -weak  $M$ -sequence if

$$((x_1, \dots, x_{i-1})M :_M x_i) \subseteq ((x_1, \dots, x_{i-1})M :_M \mathfrak{a})$$

for all  $i = 1, \dots, r$ .

Let  $x_1, \dots, x_d$  be a system of parameters for  $M$  and let  $\mathfrak{q} = (x_1, \dots, x_d)A$ . We say that  $x_1, \dots, x_d$  is a *standard system of parameters* of  $M$  if  $x_1^{n_1}, \dots, x_d^{n_d}$  is a  $\mathfrak{q}$ -weak  $M$ -sequence for all  $n_1, \dots, n_d \geq 1$ .

We say that  $\mathfrak{a}$  is a *standard ideal* with respect to  $M$  if every system of parameters of  $M$  contained in  $\mathfrak{a}$  is a standard system of parameters of  $M$ .

We begin by isolating a general statement concerning the index of reducibility of parameter ideals in the case where  $M$  has finite local cohomologies.

**Proposition 3.2.** *Let  $(A, \mathfrak{m}, k)$  be a Noetherian local ring and let  $M$  be a finitely-generated  $d$ -dimensional  $A$ -module with  $d > 0$  such that  $M$  has finite local cohomologies. There exists an integer  $\ell$  such that for every parameter ideal  $\mathfrak{q} = (x_1, \dots, x_d)$*

of  $M$ , if  $\mathfrak{q} \subseteq \mathfrak{m}^\ell$  then the index of reducibility of  $\mathfrak{q}$  on  $M$  is given by

$$(3.1) \quad N_A(\mathfrak{q}; M) = \mathfrak{S} \left( \sum_{i=1}^d \frac{U_i + x_i M}{\mathfrak{q}M} \right) + \mathfrak{S}(\mathbb{H}_{\mathfrak{m}}^d(M)),$$

where  $U_i = U((x_1, \dots, \hat{x}_i, \dots, x_d)A; M)$ .

*Proof.* Since  $M$  has finite local cohomologies, we have a standard ideal  $\mathfrak{a}$  for  $M$  [SV, Corollary 18, p. 264]; hence every system of parameters of  $M$  contained in  $\mathfrak{a}$  is an  $\mathfrak{a}$ -weak  $M$ -sequence [SV, Theorem 20, p. 264]. Thus, given any system of parameters  $x_1, \dots, x_d$  of  $M$  contained in  $\mathfrak{a}$  and any integer  $n \geq 1$ , we have by [SV, Lemma 23]

$$(3.2) \quad \begin{aligned} & ((x_1^{n+1}, \dots, x_d^{n+1})M :_M (x_1 \cdots x_d)^n) \\ &= (x_1, \dots, x_d)M + \sum_{i=1}^d ((x_1, \dots, \hat{x}_i, \dots, x_d)M :_M \mathfrak{a}). \end{aligned}$$

We note that this right hand side is equal to

$$(3.3) \quad \sum_{i=1}^d (((x_1, \dots, \hat{x}_i, \dots, x_d)M :_M x_i) + x_i M).$$

Set  $U_i = U((x_1, \dots, \hat{x}_i, \dots, x_d)A; M)$ . Since  $U_i = ((x_1, \dots, \hat{x}_i, \dots, x_d)M :_M x_i)$ , we have that

$$(3.4) \quad ((x_1^{n+1}, \dots, x_d^{n+1})M :_M (x_1 \cdots x_d)^n) = \sum_{i=1}^d (U_i + x_i M).$$

An important component of this proof is that according to [GS1, Lemma 3.12], we may choose  $\ell$  large enough so that for any ideal  $\mathfrak{q}$  generated by a system of parameters for  $M$ , if  $\mathfrak{q} \subseteq \mathfrak{m}^\ell$  then the canonical map  $\phi : M/\mathfrak{q}M \rightarrow \mathbb{H}_{\mathfrak{m}}^d(M)$  is surjective on the socles; that is,  $\text{Hom}_A(k, \phi)$  is surjective. We also require  $\ell$  to be large enough so that  $\mathfrak{m}^\ell \subseteq \mathfrak{a}$ .

Let  $\mathfrak{q} = (x_1, \dots, x_d)$  be a parameter ideal of  $M$  contained in  $\mathfrak{m}^\ell$  and set  $U_i = U((x_1, \dots, \hat{x}_i, \dots, x_d)A; M)$ . Let  $K$  denote the kernel of the canonical map  $\phi$  from  $M/\mathfrak{q}M$  to  $\mathbb{H}_{\mathfrak{m}}^d(M)$ . According to the definition of the direct limit, we have

$$(3.5) \quad K = \frac{\cup_{n \geq 1} ((x_1^{n+1}, \dots, x_d^{n+1})M :_M (x_1 \cdots x_d)^n)}{\mathfrak{q}M},$$

thus by Equation 3.4 we see that

$$(3.6) \quad K = \sum_{i=1}^d \frac{U_i + x_i M}{\mathfrak{q}M}.$$

Now all that is left is to apply the socle functor  $\text{Hom}_A(k, -)$  to the exact sequence

$$(3.7) \quad 0 \rightarrow K \rightarrow M/\mathfrak{q}M \rightarrow \mathbb{H}_{\mathfrak{m}}^d(M)$$

and use the surjectivity on the socles to obtain the result.  $\square$

**Theorem 3.3.** *Suppose  $(A, \mathfrak{m})$  is a Noetherian local ring and let  $M$  be a finitely-generated  $A$ -module of dimension 2 such that  $M$  has positive depth, and  $\mathbb{H}_{\mathfrak{m}}^1(M)$*

is finitely generated. Then there exists an integer  $\ell$  such that for every parameter ideal  $\mathfrak{q}$  of  $M$ , if  $\mathfrak{q} \subseteq \mathfrak{m}^\ell$  then the index of reducibility of  $\mathfrak{q}$  on  $M$  is given by

$$(3.8) \quad N_A(\mathfrak{q}; M) = 2 \cdot \mathfrak{S}(\mathbf{H}_\mathfrak{m}^1(M)) + \mathfrak{S}(\mathbf{H}_\mathfrak{m}^2(M)).$$

In particular, parameter ideals for  $M$  in large powers of  $\mathfrak{m}$  all have the same index of reducibility.

*Proof.* We begin by obtaining an integer  $\ell$  from Proposition 3.2. As in the proof of that proposition, we may assume that  $\ell$  is large enough so that every system of parameters of  $M$  contained in  $\mathfrak{m}^\ell$  is a standard system of parameters of  $M$ .

Let  $a, b$  be a system of parameters of  $M$  contained in  $\mathfrak{m}^\ell$  and set  $\mathfrak{q} = (a, b)A$ . Then we have

$$(3.9) \quad N_A(\mathfrak{q}; M) = \mathfrak{S}\left(\frac{U_a + bM}{\mathfrak{q}M} + \frac{U_b + aM}{\mathfrak{q}M}\right) + \mathfrak{S}(\mathbf{H}_\mathfrak{m}^2(M)),$$

where  $U_a = U(aA; M) = (aM :_M b)$ , and similarly for  $U_b$ .

Since  $M$  has positive depth,  $a$  and  $b$  are regular elements on  $M$ . According to [SV, Theorem and Definition 17, p. 261],  $a$  and  $b$  both kill  $\mathbf{H}_\mathfrak{m}^1(M)$ . Thus from the long exact sequence for local cohomology obtained from the short exact sequence

$$(3.10) \quad 0 \longrightarrow M \xrightarrow{a} M \longrightarrow M/aM \longrightarrow 0$$

we see that  $\mathbf{H}_\mathfrak{m}^0(M/aM) \cong \mathbf{H}_\mathfrak{m}^1(M)$ . Since  $M/aM$  has dimension 1,  $\mathbf{H}_\mathfrak{m}^0(M/aM)$  is just  $U(aA; M)/aM = (aM :_M b)/aM$ . Furthermore, we have a surjective homomorphism  $(aM :_M b) \rightarrow ((aM :_M b) + bM)/bM$  whose kernel is  $(aM :_M b) \cap bM$ . Using the definition of a standard system of parameters, we see that this last expression is just  $aM$ . Thus we have seen that  $\mathbf{H}_\mathfrak{m}^1(M) \cong ((aM :_M b) + bM)/bM$ . The same holds if we interchange  $a$  and  $b$ .

At this point all that remains is to see that the sum

$$(3.11) \quad \frac{U_a + bM}{\mathfrak{q}M} + \frac{U_b + aM}{\mathfrak{q}M}$$

is direct. To this end, we note that

$$(3.12) \quad \begin{aligned} (U_a + bM) \cap (U_b + aM) &= (U_a \cap (U_b + aM)) + bM \\ &= (U_a \cap U_b) + aM + bM \end{aligned}$$

Using the fact that  $a$  and  $b$  are regular on  $M$  and form a standard system of parameters of  $M$ , we see that

$$(3.13) \quad U_a \cap U_b = (aM :_M b) \cap (bM :_M a) = aM \cap bM.$$

Thus

$$(3.14) \quad (U_a + bM) \cap (U_b + aM) = \mathfrak{q}M,$$

the sum is direct, and our proof is complete.  $\square$

**Corollary 3.4.** *Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension at most 2. Then  $A$  is Gorenstein if and only if  $A$  has finite local cohomologies and every power of the maximal ideal contains an irreducible parameter ideal.*

*Proof.* If  $A$  is Gorenstein, then all the local cohomology modules other than  $\mathbf{H}_\mathfrak{m}^2(A)$  are zero, and all the parameter ideals are irreducible.

For the other direction, note that it suffices to show that  $A$  is Cohen-Macaulay, since a Cohen-Macaulay local ring with an irreducible parameter ideal is precisely a

Gorenstein local ring. The theorem is trivial in dimension zero: a zero dimensional Noetherian local ring is Gorenstein if and only if the zero ideal is irreducible. When the dimension of  $A$  is one, the result essentially goes back to Northcott-Rees [NR1, Lemma 7]: If  $A$  has SCI, then  $A$  has positive depth.

In dimension 2, using the fact that the depth is positive, we know from Theorem 3.2 that all parameter ideals in a high power of the maximal ideal have the same index of reducibility, namely  $2 \cdot \mathfrak{S}(\mathbb{H}_{\mathfrak{m}}^1(A)) + \mathfrak{S}(\mathbb{H}_{\mathfrak{m}}^2(A))$ . According to our hypothesis, this integer must be 1. Since  $\mathbb{H}_{\mathfrak{m}}^2(A)$  is a nonzero Artinian module, it has a nonzero socle. Thus it has a cyclic socle, and  $\mathbb{H}_{\mathfrak{m}}^1(A)$  is an Artinian module with zero socle. Thus  $\mathbb{H}_{\mathfrak{m}}^1(A)$  is zero, so that  $A$  is Cohen-Macaulay.  $\square$

**Remark 3.5.** The technique in this section can be used to give a different proof in the case of dimension 1.

#### 4. AN EXAMPLE

We begin with a lemma concerning idealizations.

**Definition 4.1.** Given a ring  $R$  and an  $R$ -module  $M$  we define the ring  $R \times M$  to be the symmetric algebra of  $M$  modulo the square of the positive piece. Thus  $R \times M$  is a graded ring whose degree 0 piece is  $R$ , whose degree 1 piece is  $M$ , and whose components of degree greater than 1 are zero. This ring is called the *idealization* of  $R$  with  $M$ , or the *trivial extension* of  $R$  by  $M$ .

For more information on idealization, we refer the reader to [Na, p. 18] or [BH, Exercise 3.3.22].

**Lemma 4.2.** (1) *Let  $R$  be a ring and let  $M$  be an  $R$ -module. If  $I$  is an ideal of  $R$ , then*

$$\frac{R \times M}{I(R \times M)} \cong \frac{R}{I} \times \frac{M}{IM}.$$

(2) *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and let  $M$  be an  $R$ -module. Set  $A = R \times M$ . Then*

$$\mathbb{S}(A) = \mathbb{S}(R) \cap \text{ann}_R(M) + \mathbb{S}(M).$$

*In particular, we have*

$$\mathfrak{S}(A) = \mathfrak{S}(M) + \dim_k(\mathbb{S}(R) \cap \text{ann}_R(M)).$$

(3) *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and let  $M$  be an  $R$ -module. Set  $A = R \times M$ . If  $\mathfrak{q}$  is an irreducible  $\mathfrak{m}$ -primary ideal of  $R$  which does not contain  $\text{ann}_R(M)$ , then  $\mathfrak{q}A$  is a parameter ideal of  $A$  for which*

$$N_A(\mathfrak{q}A; A) = N_R(\mathfrak{q}; M) + 1.$$

*Proof.* (1) As  $R$ -modules, we have the isomorphism, and we immediately see that this homomorphism respects multiplication.

(2) We regard  $A$  as  $A = R + Mt$  with  $t$  an indeterminate and  $t^2 = 0$ . The maximal ideal of  $A$  is  $\mathfrak{m} + Mt$ . An element  $r + mt$  is in  $\mathbb{S}(A)$  if and only if

$$0 = (r + mt)(\mathfrak{m} + Mt) = r\mathfrak{m} + (rMt + \mathfrak{m}mt),$$

which happens precisely when  $r \in \mathbb{S}(R) \cap \text{ann}_R(M)$  and  $m \in \mathbb{S}(M)$ , as desired. The statement about dimensions now follows from the fact that the  $k$ -vector space structure of the socle is induced through its graded  $R$ -module structure.

(3) We have

$$\begin{aligned}
N_A(\mathfrak{q}A; A) &= \mathfrak{S}(A/\mathfrak{q}A) \\
&= \mathfrak{S}\left(\frac{R}{\mathfrak{q}} \times \frac{M}{\mathfrak{q}M}\right) \\
&= \mathfrak{S}(M/\mathfrak{q}M) + \dim_k \mathfrak{S}(R/\mathfrak{q}) \cap \text{ann}_{R/\mathfrak{q}}(M/\mathfrak{q}M) \\
&= N_R(\mathfrak{q}; M) + \dim_k \mathfrak{S}(R/\mathfrak{q}) \cap \text{ann}_{R/\mathfrak{q}}(M/\mathfrak{q}M)
\end{aligned}$$

Now since  $\mathfrak{q}$  is an irreducible  $\mathfrak{m}$ -primary ideal, the socle of  $R/\mathfrak{q}$  is simple and essential, and is thus contained in every nonzero submodule of  $R/\mathfrak{q}$ . Hence all that remains is to note that  $\text{ann}_{R/\mathfrak{q}}(M/\mathfrak{q}M)$  is not zero, since it contains  $(\mathfrak{q} + \text{ann}_R(M))/\mathfrak{q}$ .  $\square$

We thank the referee for suggesting the following example in dimension 2, and for pointing out the interesting paper of C. Lech [L].

**Example 4.3.** Let  $k$  be a field and let  $d > 1$  be an integer. Put  $R = k[[x, y, z_3, \dots, z_d]]$ , the formal power series ring over  $k$  in  $d$  variables. Let  $M = R/x^2R$  and set  $A = R \times M$ . Let  $w$  be a new variable; then  $A \cong T/(x^2w, w^2)T$ , where  $T = k[[x, y, z_3, \dots, z_d, w]]$ . Thus  $A$  is a complete Noetherian local ring of dimension  $d$  and depth  $d - 1$ .

For each integer  $n \geq 3$  we define two parameter ideals of  $R$ :

$$\mathfrak{q} = (x^n, y^n, z_3^n, \dots, z_d^n)R \quad \text{and} \quad \mathfrak{q}' = ((x + y)^n, xy^{n-1}, z_3^n, \dots, z_d^n)R.$$

Let  $Q = \mathfrak{q}A$  and  $Q' = \mathfrak{q}'A$ ; these are parameter ideals of  $A$  contained in  $\mathfrak{m}^n$ . We show that  $N_A(Q; A) = 2$  while  $N_A(Q'; A) = 3$ .

Neither  $Q$  nor  $Q'$  contain  $\text{ann}_R(M) = x^2R$ , and since  $R$  is regular (hence Gorenstein), each of  $\mathfrak{q}$  and  $\mathfrak{q}'$  are irreducible. Hence from Part 3 of Lemma 4.2 we see that to decide the index of reducibility of  $Q$  and  $Q'$ , we just need to consider

$$N_R(\mathfrak{q}; M) = \dim_k R/(\mathfrak{q} + x^2R),$$

and similarly for  $\mathfrak{q}'$ .

We have

$$\mathfrak{q} + x^2R = (x^2, y^n, z_3^n, \dots, z_d^n)R \quad \text{and} \quad \mathfrak{q}' + x^2R = (x^2, xy^{n-1}, y^n, z_3^n, \dots, z_d^n)R.$$

The first of these ideals is a parameter ideal for the Gorenstein ring  $R$ , and is thus irreducible. For the second we have

$$(x^2, xy^{n-1}, y^n, z_3^n, \dots, z_d^n)R = (x^2, y^{n-1}, z_3^n, \dots, z_d^n)R \cap (x, y^n, z_3^n, \dots, z_d^n)R.$$

Hence  $N_A(Q; A) = 1 + N_R(\mathfrak{q}; M) = 2$  while  $N_A(Q'; A) = 1 + N_R(\mathfrak{q}'; M) = 3$ .

**Remark 4.4.** We would like to point out that a bad example such as this may be obtained among local Noetherian domains. The referee kindly brought our attention to a paper of C. Lech [L] which proves that a complete Noetherian local ring  $A$  is the completion of a Noetherian local domain if and only if the prime ring of  $A$  is a domain that acts without torsion on  $A$ , and the maximal ideal of  $A$  is not associated to 0. (Here the prime ring of a ring is the subring generated by the multiplicative identity.)

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