Smooth Continuation of Traveling Waves in Phase Field Models

It is well-known that certain traveling waves can be modeled as heteroclinic connections between equilibrium points of a dynamical system [1, 4]. In particular, in [4] the continuation of invariant subspaces algorithm from [3] is adapted to develop a general algorithm for the computation and continuation of connecting orbits.

Phase field models are continuum descriptions of phase transitions from an initial to a final state. These transitions have been studied e.g. in [7, 10, 11], where solidification processes are analyzed by computing traveling waves, and some numerical examples are provided.

The author proposes to study the problem of transition between different phases, say from liquid to solid, from the point of view of connecting orbits between equilibrium points. Following [1], the first part of the project will deal with establishing a well-posed boundary value problem. A theoretical study of existence, uniqueness and stability of solutions will then be followed by the numerical analysis and computation of the connecting orbits, to show that these correspond to the traveling waves we are looking for. Several numerical examples should illustrate the effectiveness of the approach. A first set of results in this direction has been reported in [9].

This project will be a challenging and at the same time a reasonable and exciting experience to REU participants, given the well-defined structure of the problem and the existing software and numerical algorithms available [4, 5, 6] .

The author also proposes other levels of completion. The first one would be to perform continuation of those solutions as certain parameters vary. This would lead to branches of traveling wave fronts, a topic with wide applications on its own. The second one would consider the generalization of the connections in the original problem to those involving one or more periodic traveling waves. In this new context, some other applications arise, such as the computation of solitary waves in reduced water-wave problems [2] and space mission designs [8]. At these new levels, more advanced theoretical concepts (monodromy matrices, Floquet theory, etc.), and other numerical techniques (e.g. numerical computation of smooth Schur factorizations) will be needed. This would be truly challenging problems for the most ambitious and talented students, and for the continuation of REU projects in the next years.

Prerequisites: A basic background on linear algebra and differential equations. Optimally, some programming experience is desirable, but not required.

References

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