Gaussian Quadrature Formulas

Numerical Integration

EQUATION 1

The integral from *a* to *b* is approximately the sum of n+1 products, where the *i*th product is the function evaluated at the *i*th node times a certain coefficient for 0<=i<=n.

$$\int_a^b f(x) \approx \sum_{i=0}^{n} A_i f(x_i)$$

Link to Polynomial Interpolation

$$If\ f(x) \approx p(x),\ then \int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$$

Link to Polynomial Interpolation

Lagrange interpolation Formula:

$$p(x) = \sum_{0}^{n} f(x_i) L_i(x)$$

$$L_i(x) = \frac{\prod_{j=0}^{n} (x - x_j)}{(x_i - x_j)} \quad where\ j \neq i$$

Link to Polynomial Interpolation

Lagrange's interpolation formula is exactly accurate at the nodes. That is,

$$p(x_i) = f(x_i) \qquad \text{for} \qquad 0 \leq i \leq n$$

Theorem 1

Let q be a nontrivial polynomial of degree n+1 such that

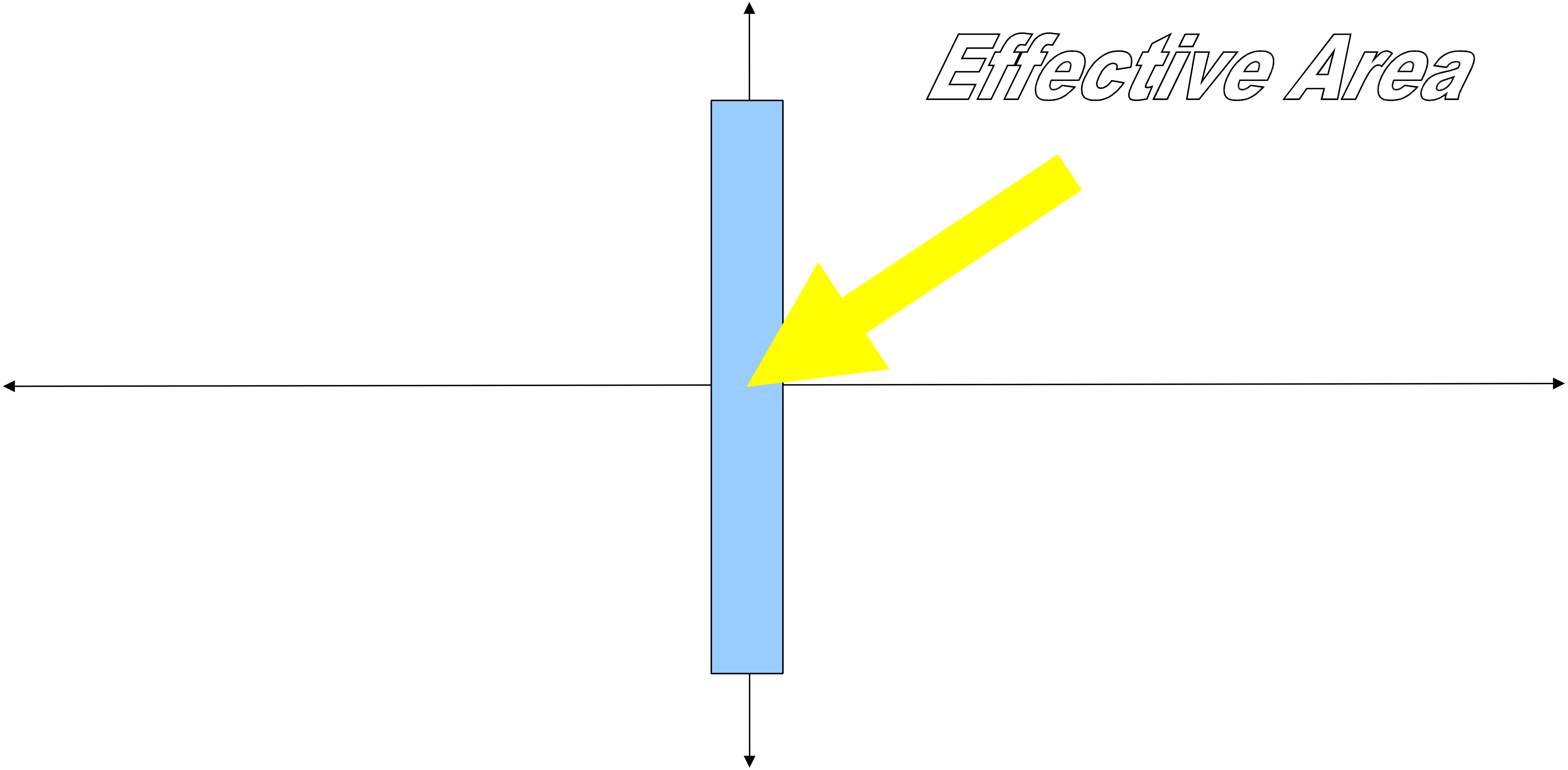
$$\int_a^b x^k q(x)\,dx = 0 \qquad (0 \le k \le n)$$

Let $x_0, x_1, \ldots, x_n$ be the zeros of q. Then the formula

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i) \qquad A_i = \int_a^b l_i(x)\,dx$$

will be exact for all polynomials of degree at most 2n+1. Furthermore, the nodes lie in the open interval (a,b).

The Bad News



Change of variables to the rescue!

$$\int_{a}^{b} f(x)dx = \frac{1}{2}(b-a)\int_{-1}^{1} f\left[\frac{1}{2}(b-a)t + \frac{1}{2}(b+a)\right]dt$$

Proof of Theorem 1

We want to show that

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

will be exact for all polynomials of degree at most 2n+1, where $x_0, x_1, \ldots, x_n$ are the nodes of q(x), an orthogonal polynomial of degree n+1, and

$$A_i = \int_a^b l_i(x)\,dx$$

Proof continued...

Let *f* be any polynomial of degree at most 2n+1. Divide *f* by *q.* This gives us a quotient function *p* and a remainder function *r*, both of which have degree at most n. In short, *f* = *pq* + *r*.
We have it given that

$$\int_a^b q(x)\,p(x)\,dx = 0.$$

We can also see that $f(x_i) = p(x_i)q(x_i) + r(x_i) = r(x_i)$ since each x_i is a root of q(x).

Proof continued...

What's more, we know that since r has degree at most n, we can obtain $\int_a^b r(x)\,dx$ precisely using the formula

$$\int_a^b r(x)\,dx = \sum_{i=0}^{n} A_i r(x_i).$$

Thus,

$$\int_a^b f(x)dx = \int_a^b p(x)q(x)\,dx + \int_a^b r(x)\,dx = \int_a^b r(x)\,dx = \ldots$$

$$\ldots = \sum_{i=0}^{n} A_i r(x_i) = \sum_{i=0}^{n} A_i f(x_i).$$

Enter Legendre or Mind your p's and q's. Especially q's

$$q_0(x) = 1$$

$$q_1(x) = x$$

$$q_2(x) = \left(\frac{3}{2}\right)x^2 - \frac{1}{2}$$

$$q_3(x) = \left(\frac{5}{2}\right)x^3 - \left(\frac{3}{2}\right)x$$

In General, Legendre Polynomials:

$$q_n(x) = \frac{(2\mathrm{n}-1)}{n} x q_{(n-1)}(x) - \frac{(n-1)}{n} q_{(n-2)}(x)$$