The Genus of Zero Divisor Graphs

Throughout, all rings are assumed to be commutative rings with identity. The *zero divisor graph* of a ring is the (simple) graph whose vertex set is the set of nonzero zero divisors, and an edge is drawn between two distinct vertices if their product is zero. This definition is the same as that introduced in [4].

In recent years, the study of zero divisor graphs has grown in various directions. At the heart is the interplay between the ring theoretic properties of a ring *R* and the graph theoretic properties of its zero divisor graph, begun in [4] and continued in [3], [2].

A ring is called planar if its zero divisor graph can be drawn in a plane so that its edges intersect only at their common vertices. Similarly, a ring is called toroidal if its zero divisor graph can be drawn on a torus so that its edges intersect only at their common vertices. In [3], the question was asked: For which finite commutative rings have a planar zero divisor graph? A partial answer was given in [1], but the question remained open for local rings of order 32. In [9] and independently in [5] it was shown that no local ring of order 32 has a planar zero-divisor graph. In [11] a complete list of finite toroidal rings was given.

In this project, we will further explore the question of embedding a zero-divisor graph on surfaces of higher genus.

Let *q* be a prime power and define two numbers $g(q)$ and $G(q)$ as follows.

$$g(q) = \min\left\{\gamma\left(\Gamma\left(R\right)\right) \mid R \text{ is a local ring, not a field, with } |R| = q\right\}$$

and

$$G(q) = \max\left\{\gamma\left(\Gamma\left(R\right)\right) \mid R \text{ is a local ring, not a field, with } |R| = q\right\}.$$

We will investigate the numbers $g(q)$ and $G(q)$ in this project, and answer as best as possible the following questions. Clearly, $g(q) \le G(q)$ for all prime powers *q*.

Question 1. Is $g\left(p^i\right) < G\left(p^i\right)$ for odd primes *p* and integers $i > 2$, and $g\left(2^i\right) < G\left(2^i\right)$ for integers $i > 3$?

[11, Theorem 4.1] (and its proof) shows that $0 = g(27) < G(27) = 2$, and that $g\left(2^i\right) < G\left(2^i\right)$ for $i = 4,5,6$, giving an affirmative answer to Question 1 in these cases.

Question 2. If $|R| = q$, $g(q) < G(q)$, and $\gamma\left(\Gamma\left(R\right)\right) = g(q)$, is *R* Gorenstein?

[11, Theorem 4.1] (and its proof) gives an affirmative answer to Question 2 if $q = 16, 27, 32, 64$.

Prerequisites: One semester of abstract algebra. Some exposure to graph theory is desirable, but not required.

References

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