The Finite Difference Method for the Helmholtz Equation with Applications to Cloaking

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Abstract

Many recent papers have focused on the theoretical construction of “cloaking devices” which have the capability of shielding an object from observation by electromagnetic waves. The goal of this work is to visualize the effect these theoretical cloaking devices have on traveling waves. We apply the finite difference method to determine numerical solutions of boundary value problems for a specific generalized version of the Helmholtz equation and plot the results to illustrate how waves can theoretically be redirected around an object in order to hide it from an outside observer.

1 Introduction

In the past few years, scientists have made great progress in the field of cloaking. Cloaking involves making an object invisible or undetectable to electromagnetic waves. Recent papers such as [2], [3], and [5] have dealt with improving theoretical cloaking techniques based on singular transformations of the domains through which the waves travel. In this paper, we use the finite difference method to approximate the partial differential equations that model wave propagation, which allows us to plot numerical solutions in order to visualize the effect that these cloaking constructions can have on a traveling wave. The construction we focus on is referred to in [3] as the single coating construction, and it leads to the discussion of divergence form operators which can be used to model a wave traveling through an inhomogeneous medium. We consider solutions to a generalized Helmholtz equation in two dimensions and show how the corresponding waves can bend around a given region and still emerge on the other side as if the waves had passed through empty space, thus rendering the region “invisible” to outside detection. We include specific examples to illustrate how two different waves could appear indistinguishable to an outside observer, providing both numerical and visual support for the theory of cloaking.

2 The Helmholtz Equation

2.1 Derivation from the Wave Equation

The wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \Delta w \tag{2.1}$$

models the propagation of a wave travelling through a given medium at a constant speed c. Here Δ is the Laplacian, which in two dimensional space is given by $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. If we assume the solution w is separable, then we can write $w(x, y, t) = u(x, y)v(t)$. Since u is independent of t and v is independent of x and y, substituting this into (2.1) then gives

$$u\frac{\partial^2 v}{\partial t^2} = c^2 v\Delta u, \tag{2.2}$$

which can be rewritten as

$$\frac{1}{c^2 v}\frac{\partial^2 v}{\partial t^2} = \frac{\Delta u}{u}. \tag{2.3}$$

Since the left side of (2.3) is a function of t and the right side is a function of x and y, the only way the two sides can be equal is if both functions are constant. If we assume both sides are equal to the constant $-s^2$, then solving (2.1) reduces to solving the two equations

$$\frac{\partial^2 v}{\partial t^2} + c^2 s^2 v = 0, \tag{2.4}$$

and

$$\Delta u + s^2 u = 0. \tag{2.5}$$

Equation (2.4) has solutions of the form

$$v(t) = c_1\cos(\omega t) + c_2\sin(\omega t), \text{ where } \omega = cs. \tag{2.6}$$

Equation (2.5) is referred to as the Helmholtz equation and will be the focus of this paper.

2.2 Approximating Derivatives

To apply the finite difference method, we will need to estimate the derivatives of a function based only on the function's values at specific points. If $f \in C^\infty(\mathbb{R})$, then from Taylor's formula,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots, \tag{2.7}$$

and similarly,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(x) + \ldots. \tag{2.8}$$

Subtracting then gives

$$f(x+h) - f(x-h) = 2hf'(x) + \mathcal{O}(h^3), \tag{2.9}$$

and so we can approximate the derivative of f using the formula

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}, \tag{2.10}$$

with error term $\mathcal{O}(h^2)$. Furthermore, adding (2.7) and (2.8) gives

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \mathcal{O}(h^4), \tag{2.11}$$

which gives the approximation

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}, \tag{2.12}$$

with error term $\mathcal{O}(h^2)$. In a similar fashion to (2.10) and (2.12), if $u \in C^\infty(\mathbb{R}^2)$, we can approximate the partial derivatives of f as follows

$$\frac{\partial u}{\partial x}(x,y) \approx \frac{u(x+h,y) - u(x-h,y)}{2h}, \tag{2.13}$$

$$\frac{\partial u}{\partial y}(x,y) \approx \frac{u(x,y+k) - u(x,y-k)}{2k}, \tag{2.14}$$

$$\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}, \tag{2.15}$$

$$\frac{\partial^2 u}{\partial y^2}(x,y) \approx \frac{u(x,y+k) - 2u(x,y) + u(x,y-k)}{k^2}. \tag{2.16}$$

2.3 The Finite Difference Method

Let $R = [a,b] \times [c,d]$ be a rectangle in $\mathbb{R}^2$, and consider the boundary value problem

$$\begin{cases} \Delta u + s^2 u = 0 & \text{in } R, \\ u(x,y) = f(x,y) & \text{on } \partial R. \end{cases} \tag{2.17}$$

To apply the finite difference method to this problem, we first approximate R with an $n \times m$ lattice of the form (x_i, y_j) where

$$x_i = a + (i-1)h, \quad h := \frac{b-a}{n-1}, \quad \text{and} \quad y_j = c + (j-1)k, \quad k := \frac{d-c}{m-1}. \tag{2.18}$$

For a given lattice point (x,y), we can then approximate the equation $\Delta u + s^2 u = 0$ using (2.15) and (2.16) to derive the equation

$$\frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2} + \frac{u(x,y+k) - 2u(x,y) + u(x,y-k)}{k^2} + s^2 u(x,y) = 0. \tag{2.19}$$

Letting $r = \frac{k^2}{h^2}$, this then simplifies to the following five-point formula,

$$ru(x+h,y) + ru(x-h,y) + u(x,y+k) + u(x,y-k) + \left(s^2k^2 - 2r - 2\right) u(x,y) = 0. \tag{2.20}$$

So for each lattice point (x,y), we can write a similar linear equation involving the values of u at nearby points. Since the values of u are known along the boundary and unknown at the $(n-2)(m-2)$ interior lattice points, this gives us a linear system of $(n-2)(m-2)$ equations and $(n-2)(m-2)$ variables to solve. Let us enumerate our $(n-2)(m-2)$ variables by z_k, where

$$z_k = u(x_i, y_j) \quad \text{for} \quad k = (j-2)(n-2) + i - 1. \tag{2.21}$$

This allows us to express our system of linear equations in the form $A\vec{z} = \vec{b}$, where A is an $(n-2)(m-2) \times (n-2)(m-2)$ matrix. Solving the corresponding system gives an estimate of u at each of the interior lattice points, which provides us with a numerical solution for (2.17).

A MATLAB program was written to execute the algorithm above to solve (2.17). After constructing the lattice and determining the appropriate matrix A, the program solves the linear system

$A\vec{z} = \vec{b}$ using the Gauss-Seidel method as described in [1]. The solution u can then be plotted. For example, let R be the square $[-3, 3] \times [-3, 3]$, and consider the boundary value problem

$$\begin{cases} \Delta u + u = 0 & \text{in } R, \\ u(x, y) = \sin\left(\dfrac{\pi x}{6}\right) & \text{on } \partial R. \end{cases} \tag{2.22}$$

Using a 70×70 lattice resulted in the solution pictured in Figure 1. Similarly, the solution to

$$\begin{cases} \Delta u + u = 0 & \text{in } R, \\ u(x, y) = \dfrac{-1}{x^2 + y^2} & \text{on } \partial R. \end{cases} \tag{2.23}$$

is pictured in Figure 2.

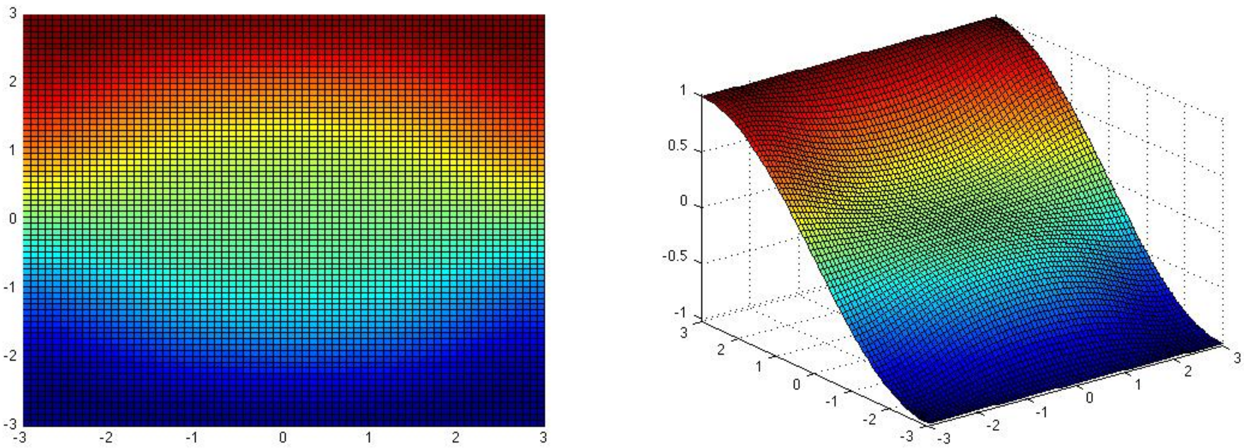


Figure 1: The solution of (2.22) on $R = [-3, 3] \times [-3, 3]$.

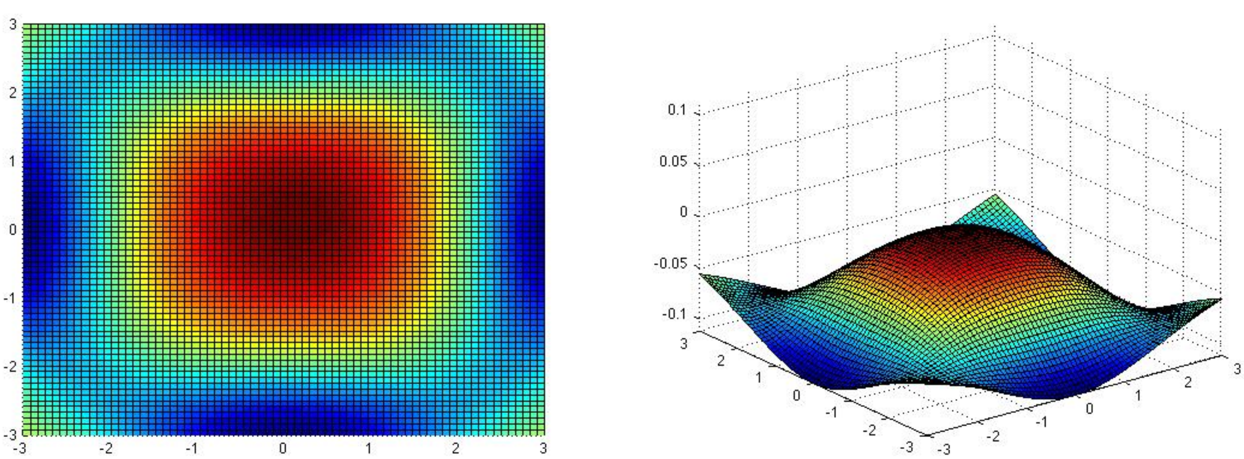


Figure 2: The solution of (2.23) on $R = [-3, 3] \times [-3, 3]$.

3 Divergence Form Operators

3.1 Transformations

A *divergence form* operator acting on functions $u \in C^2(\mathbb{R}^2)$ is a differential operator L of the form

$$Lu = \operatorname{div} A\nabla u, \tag{3.1}$$

where $A = A(x,y)$ is a 2×2 matrix, and ∇u is the gradient of u. Note that in the case when A is the identity matrix, then $Lu = \operatorname{div} \nabla u = \Delta u$. Divergence form operators of this type arise in a variety of situations. These operators are better suited for modeling the case of a wave traveling through an inhomogeneous medium, where the path of the wave is altered by the varying physical properties of the domain. The matrix $A = A(x,y)$, in this case, varies as well to model the change in these physical properties throughout the domain.

A related example occurs when a domain R is transformed into a new domain $\tilde{R}$. In this case, solving $\Delta u = 0$ on R turns out to be equivalent to solving a problem of the form $Lu = 0$ on $\tilde{R}$. To explain this relationship further, we first need a few definitions. First, for a differentiable function $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, define the matrix

$$DF(x,y) = \begin{bmatrix} \dfrac{\partial F_1}{\partial x}(x,y) & \dfrac{\partial F_1}{\partial y}(x,y) \\ \dfrac{\partial F_2}{\partial x}(x,y) & \dfrac{\partial F_2}{\partial y}(x,y) \end{bmatrix}, \quad \text{where} \quad F(x,y) = (F_1(x,y), F_2(x,y)).$$

For $\Omega \subset \mathbb{R}^2$, let $C_o^\infty(\Omega)$ denote the set of smooth functions $\varphi \in C^\infty(\mathbb{R}^2)$ that are equal to zero outside of some compact set $K \subset \Omega$. It follows if $u \in C^2(\Omega)$, then

$$Lu = 0 \text{ in } \Omega \quad \Longleftrightarrow \quad \int_\Omega Lu\, \varphi\, dx = 0, \quad \forall \varphi \in C_o^\infty(\Omega). \tag{3.2}$$

Using this result, we can establish the following.

Lemma 3.1 *Let $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^2$, and assume $F : \tilde{\Omega} \longrightarrow \Omega$ is an isomorphism such that F and F^{-1} are smooth functions. Let $A = A(x,y)$ be the 2×2 matrix given by*

$$A(x,y) = \frac{1}{|\det DF(\tilde{x},\tilde{y})|}[DF(\tilde{x},\tilde{y})][DF(\tilde{x},\tilde{y})]^\top\Big|_{(\tilde{x},\tilde{y})=F^{-1}(x,y)}. \tag{3.3}$$

Define $Lu = \operatorname{div} A\nabla u$. Then if $u \in C^2(\Omega)$, we have

$$Lu = 0 \text{ in } \Omega \iff \Delta(u \circ F) = 0 \text{ in } \tilde{\Omega}.$$

Proof. Let $\tilde{u} = u \circ F$. After integrating by parts in (3.2), it is enough to show that

$$\int_{\tilde{\Omega}} \nabla\tilde{u} \cdot \nabla\tilde{\varphi}\, d\tilde{x}\, d\tilde{y} = 0, \quad \forall \tilde{\varphi} \in C_o^\infty(\tilde{\Omega}) \iff \int_\Omega A\nabla u \cdot \nabla\varphi\, dx\, dy = 0, \quad \forall \varphi \in C_o^\infty(\Omega). \tag{3.4}$$

For a given $\varphi \in C_o^\infty(\Omega)$, let $\tilde{\varphi} = \varphi \circ F$. From the chain rule, we have

$$\nabla\tilde{u} = \nabla(u \circ F) = (DF)^\top[(\nabla u) \circ F].$$

It follows that

$$\int_{\tilde\Omega} \nabla\tilde u \cdot \nabla\tilde\varphi \, d\tilde x \, d\tilde y = \int_{\tilde\Omega} (DF)^\top[(\nabla u)\circ F] \cdot (DF)^\top[(\nabla\varphi)\circ F] \, d\tilde x \, d\tilde y$$

$$= \int_{\tilde\Omega} \frac{1}{|\det DF|}(DF)(DF)^\top[(\nabla u)\circ F] \cdot [(\nabla\varphi)\circ F] |\det DF| \, d\tilde x \, d\tilde y \quad (3.5)$$

Making the change of variables $(x, y) = F(\tilde x, \tilde y)$ then leads to

$$\int_{\tilde\Omega} \nabla\tilde u \cdot \nabla\tilde\varphi \, d\tilde x \, d\tilde y = \int_{\tilde\Omega} A\nabla u \cdot \nabla\varphi \, dx \, dy, \quad (3.6)$$

which is enough to verify (3.4). □

The previous lemma illustrates how problems involving divergence-from operators arise naturally when considering transformations of problems involving the Laplacian. Applications of this type will be discussed in Section 4.

3.2 A Generalization of the Helmholtz Equation

In this section, we are interested in a general Helmholtz equation of the form

$$Lu + s^2 u = 0. \quad (3.7)$$

Given a 2×2 matrix A, we wish to apply the finite difference method to solve a boundary value problem

$$\begin{cases} Lu + s^2 u = 0 & \text{in } R, \\ u(x, y) = f(x, y) & \text{on } \partial R, \end{cases} \quad (3.8)$$

where $R = [a, b] \times [c, d]$ as before, and $Lu = \text{div}A\nabla u$. For simplicity, let $\partial_1 u = \frac{\partial u}{\partial x}$ and $\partial_2 u = \frac{\partial u}{\partial y}$. If we assume

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

we can write

$$A\nabla u = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \partial_1 u \\ \partial_2 u \end{bmatrix} = \begin{bmatrix} a_{11}\partial_1 u + a_{12}\partial_2 u \\ a_{21}\partial_1 u + a_{22}\partial_2 u \end{bmatrix} \quad (3.9)$$

Then we can express Lu as

$$Lu = \text{div}A\nabla u = \sum_{i=1}^{2}\sum_{j=1}^{2} \partial_i(a_{ij}\partial_j u) = \sum_{i=1}^{2}\sum_{j=1}^{2} (\partial_i a_{ij})\partial_j u + \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}(\partial_i\partial_j u). \quad (3.10)$$

If we then let $b_j = \sum_{i=1}^{2} \partial_i a_{ij}$, we can rewrite Lu as

$$
\begin{aligned}
Lu &= \sum_{j=1}^{2} b_j \partial_j u + \sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}(\partial_i \partial_j u) \\
&= b_1\partial_1 u + b_2 \partial_2 u + a_{11}\partial_1^2 u + (a_{12}+a_{21})\partial_1\partial_2 u + a_{22}\partial_2^2 u. \qquad (3.11)
\end{aligned}
$$

To apply the finite difference method, we will use the approximations (2.13) - (2.16) in (3.11), but we will also need an approximation for $\partial_1\partial_2 u$. In the same manner as in (2.13), using (2.14) we can show that

$$
\begin{aligned}
\partial_1(\partial_2 u)(x,y) &\approx \frac{\partial_2 u(x+h,y) - \partial_2 u(x-h,y)}{2h} \\
&\approx \frac{1}{2h}\left(\frac{u(x+h,y+k)-u(x+h,y-k)}{2k} - \frac{u(x-h,y+k)-u(x-h,y-k)}{2k}\right) \\
&\approx \frac{u(x+h,y+k)-u(x+h,y-k)-u(x-h,y+k)+u(x-h,y-k)}{4hk}. \qquad (3.12)
\end{aligned}
$$

Utilizing (2.13) - (2.16) along with (3.12) in (3.11) allows us to approximate the equation $Lu + s^2 u = 0$ with the nine-point formula

$$
\begin{aligned}
&\left(\frac{a_{12}+a_{21}}{4hk}\right) u(x+h,y+k) + \left(\frac{a_{22}}{k^2}+\frac{b_2}{2k}\right) u(x,y+k) - \left(\frac{a_{12}+a_{21}}{4hk}\right) u(x-h,y+k) \\
&+ \left(\frac{a_{11}}{h^2}+\frac{b_1}{2h}\right) u(x+h,y) + \left(s^2 - \frac{2a_{11}}{h^2} - \frac{2a_{22}}{k^2}\right) u(x,y) + \left(\frac{a_{11}}{h^2}-\frac{b_1}{2h}\right) u(x-h,y) \qquad (3.13) \\
&- \left(\frac{a_{12}+a_{21}}{4hk}\right) u(x+h,y-k) + \left(\frac{a_{22}}{k^2}-\frac{b_2}{2k}\right) u(x,y-k) + \left(\frac{a_{12}+a_{21}}{4hk}\right) u(x-h,y-k) = 0.
\end{aligned}
$$

This is similar to formula (2.20), although for a general divergence form operator, more points are necessary. For each lattice point (x,y), we can also write a similar linear equation involving the values of $u(x,y)$ at nearby points. This gives a system of linear equations that can be solved as before to generate a numerical solution to (3.8). Examples will follow in the next section.

4 Applications to Cloaking

4.1 A Specific Example

In this section, we seek to solve the boundary value problem (3.8) where the divergence-from operator $Lu = \mathrm{div} A\nabla u$ is one associated with a transformation F as discussed in Section 3.1. Consider the transformation

$$
F(x,y) = \left(\left(1+\frac{\sqrt{x^2+y^2}}{2}\right)\frac{x}{\sqrt{x^2+y^2}}, \left(1+\frac{\sqrt{x^2+y^2}}{2}\right)\frac{y}{\sqrt{x^2+y^2}}\right). \qquad (4.1)
$$

Note that this transformation is the same one considered in [5] and used in [3] for the single coating construction. In order find a numerical solution for (3.8), we first have to determine the corresponding matrix A, as defined in (3.3). For simplicity, let $r = \sqrt{x^2 + y^2}$, so that F can be written as

$$F(x, y) = \left(\left(1 + \frac{r}{2}\right)\frac{x}{r}, \left(1 + \frac{r}{2}\right)\frac{y}{r}\right) = (F_1(x, y), F_2(x, y)). \tag{4.2}$$

To detemine A, we first need to compute the matrix

$$DF(x, y) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}.$$

Notice that $\partial_1 r = \dfrac{x}{r}$ and $\partial_2 r = \dfrac{y}{r}$. Then we can write

$$\frac{\partial F_1}{\partial x} = \frac{x}{r}\partial_1\left(1 + \frac{r}{2}\right) + \left(1 + \frac{r}{2}\right)\partial_1\left(\frac{x}{r}\right) = \frac{x^2}{2r^2} + \left(1 + \frac{r}{2}\right)\left(\frac{1}{r} - \frac{x^2}{r^3}\right) = \frac{2+r}{2r} - \frac{x^2}{r^3},$$

and in a similar manner, $\dfrac{\partial F_2}{\partial y} = \dfrac{2+r}{2r} - \dfrac{y^2}{r^3}$. We also have

$$\frac{\partial F_2}{\partial x} = \frac{y}{r}\partial_1\left(1 + \frac{r}{2}\right) + \left(1 + \frac{r}{2}\right)\partial_1\left(\frac{y}{r}\right) = \frac{xy}{2r^2} + \left(1 + \frac{r}{2}\right)\left(-\frac{xy}{r^3}\right) = -\frac{xy}{r^3},$$

and similarly, $\dfrac{\partial F_1}{\partial y} = -\dfrac{xy}{r^3}$. Together we have

$$DF(x, y) = \begin{bmatrix} \frac{2+r}{2r} - \frac{x^2}{r^3} & -\frac{xy}{r^3} \\ -\frac{xy}{r^3} & \frac{2+r}{2r} - \frac{y^2}{r^3} \end{bmatrix}, \tag{4.3}$$

and furthermore,

$$\det DF(x, y) = \left(\frac{2+r}{2r} - \frac{x^2}{r^3}\right)\left(\frac{2+r}{2r} - \frac{y^2}{r^3}\right) - \left(-\frac{xy}{r^3}\right)\left(-\frac{xy}{r^3}\right) = \frac{2+r}{4r}. \tag{4.4}$$

Combining (4.3) and (4.4), it can be shown that

$$\frac{(DF(\tilde{x}, \tilde{y}))(DF(\tilde{x}, \tilde{y}))^T}{|\det DF(\tilde{x}, \tilde{y})|} = \frac{4\tilde{r}}{2+\tilde{r}} \begin{bmatrix} \left(\frac{2+\tilde{r}}{2\tilde{r}}\right)^2 - \frac{(1+\tilde{r})\tilde{x}^2}{\tilde{r}^4} & -\frac{(1+\tilde{r})\tilde{x}\tilde{y}}{r^4} \\ -\frac{(1+\tilde{r})\tilde{x}\tilde{y}}{\tilde{r}^4} & \left(\frac{2+\tilde{r}}{2\tilde{r}}\right)^2 - \frac{(1+\tilde{r})\tilde{y}^2}{\tilde{r}^4} \end{bmatrix}, \tag{4.5}$$

where $\tilde{r} = \sqrt{\tilde{x}^2 + \tilde{y}^2}$. To compute the matrix $A(x, y)$ as in (3.3), we need to evaluate (4.5) at the point $(\tilde{x}, \tilde{y}) = F^{-1}(x, y)$. It can be easily verified that

$$F^{-1}(x, y) = \left(2(r-1)\frac{x}{r}, 2(r-1)\frac{y}{r}\right), \tag{4.6}$$

and if $(\tilde{x}, \tilde{y}) = F^{-1}(x, y)$, then $\tilde{r} = 2(r-1)$. Making these substitutions in (4.5) allows us to conclude that

$$A(x,y) = \begin{bmatrix} \dfrac{r}{r-1} - \dfrac{(2r-1)x^2}{(r-1)r^3} & -\dfrac{(2r-1)xy}{(r-1)r^3} \\ -\dfrac{(2r-1)xy}{(r-1)r^3} & \dfrac{r}{r-1} - \dfrac{(2r-1)y^2}{(r-1)r^3} \end{bmatrix}. \tag{4.7}$$

This will allow us to use the nine-point formula (3.13) to construct a numerical solution of (3.8) once we also determine the values of b_1 and b_2. Then since

$$\partial_1 a_{11} = \frac{(6r^2 - 8r + 3)x^3 - (5r^4 - 6r^3 + 2r^2)x}{r^5(r-1)^2}, \tag{4.8}$$

and

$$\partial_2 a_{21} = \frac{(6r^2 - 8r + 3)xy^2 - (2r^4 - 3r^3 + r^2)x}{r^5(r-1)^2}, \tag{4.9}$$

it follows that

$$b_1 = \partial_1 a_{11} + \partial_2 a_{21} = \frac{-x}{r^2(r-1)}, \tag{4.10}$$

and by a similar calculation,

$$b_2 = \partial_1 a_{12} + \partial_2 a_{22} = \frac{-y}{r^2(r-1)}. \tag{4.11}$$

Together with (4.7) and (3.13), we can now use the finite difference method to solve boundary value problems similar to (3.8). Note that, by design, the matrix $A(x,y)$ is actually singular when $r = 1$, so it would not make sense to consider the differential operator $Lu = \operatorname{div} A \nabla u$ in a domain containing the unit circle $r = 1$. We will instead will focus on the following differential operator:

$$Lu(x,y) = \begin{cases} \Delta u(x,y) & r \geq 2, \\ \operatorname{div} A \nabla u(x,y) & 1 < r < 2, \\ \Delta u(x,y) & r \leq 1, \end{cases} \tag{4.12}$$

where $A = A(x,y)$ is as in (4.7). The operator L can also be thought of as a divergence form operator associated with a matrix $\tilde{A}$, where $\tilde{A}(x,y) = A(x,y)$ for $1 < r < 2$ and $\tilde{A}(x,y) = I$ otherwise. We seek to find a numerical solution to the generalized Helmholtz equation (3.8) with this choice of L. For example, let R be the square $[-3,3] \times [-3,3]$, and consider

$$\begin{cases} Lu + u = 0 & \text{in } R, \\ u(x,y) = \sin\left(\dfrac{\pi x}{6}\right) & \text{on } \partial R, \end{cases} \tag{4.13}$$

with L as in (4.12). A MATLAB program similar to the previous one was created to apply the finite difference method to solve this boundary value problem. For lattice points (x, y) such that $1 < r < 2$, the nine-point formula (3.13) was used in place of (2.20). This again led to system of equations that could be solved to give a numerical solution. The numerical solution to (4.13) is shown in Figure 3, and should be compared to the solution of the classic Helmholtz equation given in Figure 1. Consider also the boundary value problem

$$\begin{cases} Lu + u = 0 & \text{in } R, \\ u(x,y) = \dfrac{-1}{x^2+y^2} & \text{on } \partial R. \end{cases} \tag{4.14}$$

The numerical solution of (4.14) is likewise shown in Figure 4, and should be compared with the solution pictured in Figure 2.

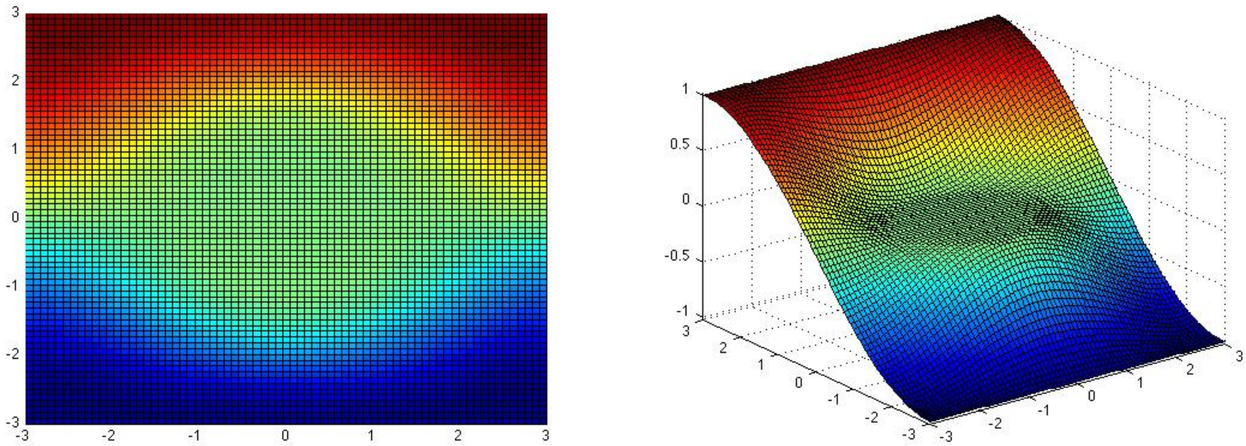


Figure 3: The solution of (4.13) on $R = [-3, 3] \times [-3, 3]$.

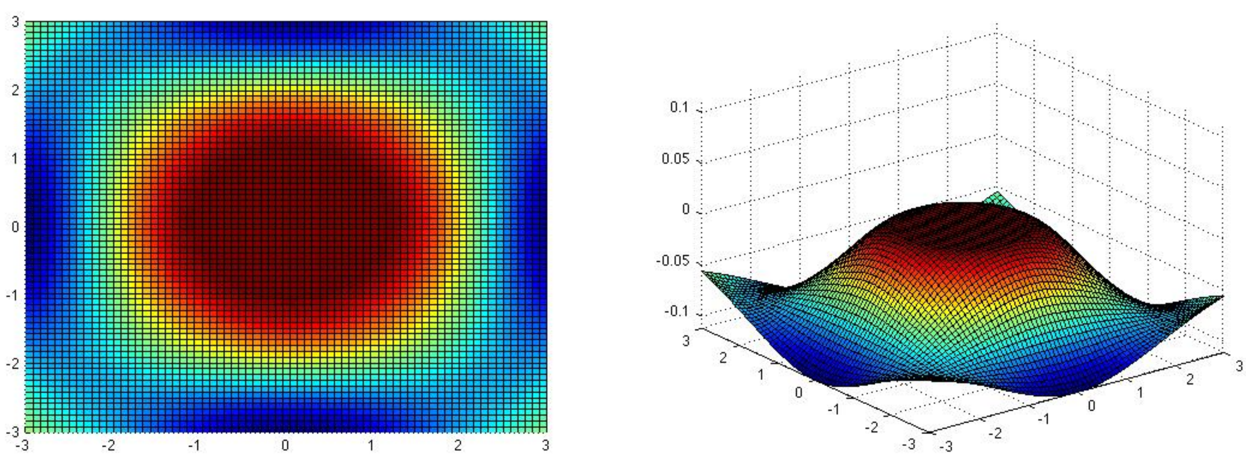


Figure 4: The solution of (4.14) on $R = [-3, 3] \times [-3, 3]$.

4.2 Indistinguishable Waves

For the rest of this section, we will refer to a function u as a wave if u satisfies the generalized Helmholtz equation $Lu + s^2u = 0$ in a domain Ω for some divergence form operator L. We will also call two waves *indistinguishable* if both their boundary values and their normal derivatives are identical. Recall that the normal derivative $\frac{\partial u}{\partial \nu}$ of a function u is the directional derivative at a point on the boundary in the direction of the outer unit normal. In other words,

$$u \text{ and } \tilde{u} \text{ are indistinguishable} \iff \frac{\partial u}{\partial \nu} = \frac{\partial \tilde{u}}{\partial \nu} \text{ and } u = \tilde{u} \text{ on } \partial\Omega.$$

Two indistinguishable waves would appear identical to an observer taking measurements along the boundary of the domain. The existence of distinct indistinguishable waves makes it impossible to determine a wave uniquely based on its boundary measurements, and furthermore makes it impossible to uniquely determine the characteristics of the domain itself (i.e., the matrix A which defines L) based on observations along the boundary. It has previously been established in [4] and mentioned in [3], that if a transformation $F : \Omega \longrightarrow \Omega$ fixes the boundary of Ω, and if $Lu = \mathrm{div} A\nabla u$, where A is as in (3.3), then the solutions u and $\tilde{u}$ of

$$\begin{cases} Lu + s^2u = 0 & \text{in } \Omega, \\ u(x,y) = f(x,y) & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\tilde{u} + s^2\tilde{u} = 0 & \text{in } \Omega, \\ \tilde{u}(x,y) = f(x,y) & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

are indistinguishable. This allows us to conclude that the wave pictured in Figure 3 would be indistinguishable from the wave in Figure 1 to an outside observer, and likewise the wave in Figure 4 would be indistinguishable from the one in Figure 2. To verify that our numerical solutions represent indistinguishable waves, we only need to compare their normal derivatives, since their boundary values are identical by construction. Since our domain is the rectangle R the outer unit normal at each point on the boundary is actually one of $\pm\frac{\partial u}{\partial x}$ or $\pm\frac{\partial u}{\partial y}$, and so these can be estimated easily using finite differences as in (2.13) and (2.14). Since our numerical solutions are only approximations, these normal derivatives will not be identical, but they should become closer together if we improve our approximations by increasing the number of lattice points used. The following table shows the average differences between the normal derivatives of the solutions of (2.22) and (4.13) and between the normal derivatives of the solutions of (2.23) and (4.14) as the size of the lattice was gradually increased from 30×30 to 80×80.

	30×30	40×40	50×50	60×60	70×70	80×80
$u, \tilde{u}$ as in (2.22) and (4.13)	0.377	0.214	0.116	0.062	0.034	0.018
$u, \tilde{u}$ as in (2.23) and (4.14)	0.161	0.044	0.018	0.009	0.005	0.003

Figure 5: Average value of $\left|\frac{\partial u}{\partial \nu} - \frac{\partial \tilde{u}}{\partial \nu}\right|$ using an $n \times n$ lattice

The data contained in Figure 5 seems to support the conclusion that the waves considered are indistinguishable.

The transformation F in (4.1) was chosen purposefully so that the waves pictured in Figures 3 and 4 would "bend" around the unit disc. In fact, if the definition of L in (4.12) were altered for $r \leq 1$, the resulting waves would still be indistinguishable from the previous ones. In particular, with this construction, it would be impossible for an outside observer to determine what actually occurs inside the unit disc, in a sense rendering the unit disc as hidden or "cloaked." If the wave in question represents a light wave, this would make it theoretically possible to bend light waves around an object in such a way that the object would not be visible to an outside observer, nor would the observer notice any difference than if the object were not there. The purpose of this paper is only to give numerical support for the possibility of cloaking. To achieve such cloaking in practice would require constructing a domain with very specific physical properties that mirror the transformation given by (4.1). Some progress in this respect has been made in [6] using artificially structured metamaterials to cloak a small copper cylinder under a narrow frequency band.

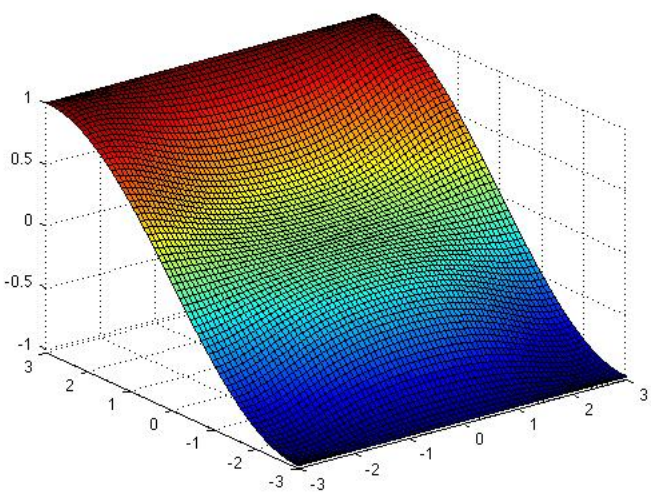
To conclude, let us return our attention to the wave equation (2.1). Using our previous results, we can now construct numerical solutions from the wave equation as described in Section 2.1. Let $R = [-3, 3] \times [-3, 3]$ as before and fix $T > 0$. Now consider the boundary value problems,

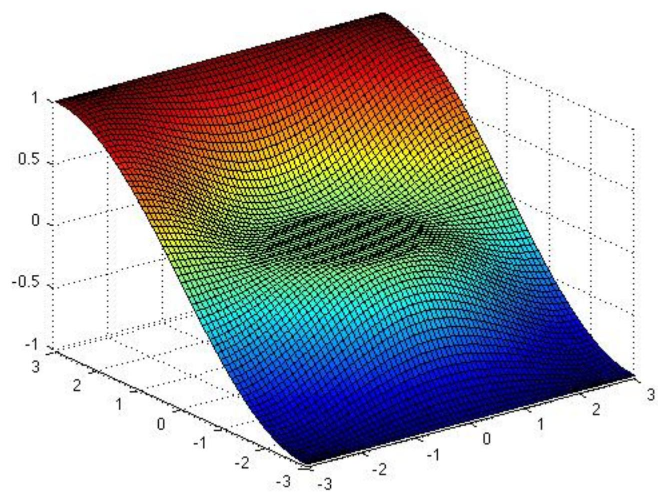
$$\begin{cases} \frac{\partial^2 w}{\partial t^2} = \Delta w \quad \text{ in } R \times [0, T], \\ w(x, y, 0) = \sin\left(\dfrac{\pi x}{6}\right) \text{ on } \partial R, \\ \dfrac{\partial w}{\partial t}(x, y, 0) = 0 \text{ in } R. \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial^2 \tilde{w}}{\partial t^2} = L\tilde{w} \quad \text{ in } R \times [0, T], \\ \tilde{w}(x, y, 0) = \sin\left(\dfrac{\pi x}{6}\right) \text{ on } \partial R, \\ \dfrac{\partial \tilde{w}}{\partial t}(x, y, 0) = 0 \text{ in } R. \end{cases} \tag{4.16}$$

By assuming the solutions are separable, we can construct solutions by first solving (2.22) and (4.13) as well as the initial value problem

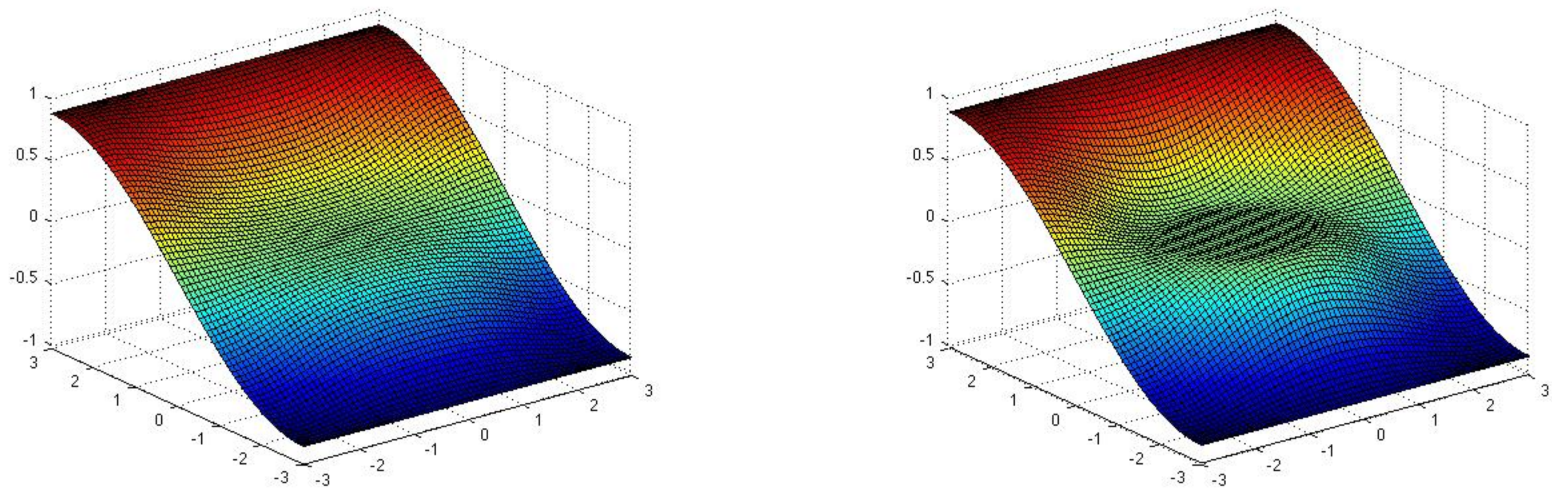
$$\begin{cases} \frac{\partial^2 v}{\partial t^2} + v = 0 \text{ in } [0, T], \\ v(0) = 1, \quad \dfrac{\partial v}{\partial t}(0) = 0, \end{cases} \tag{4.17}$$

as outlined in Section 2.1. Here we are also assuming $s = 1$. The solution to (4.17) is $v(t) = \cos t$, and the solutions to (2.23) and (4.14) are as pictured in Figures 1 and 3 respectively. The resulting numerical solutions for (4.16) at various points in time are illustrated as follows.

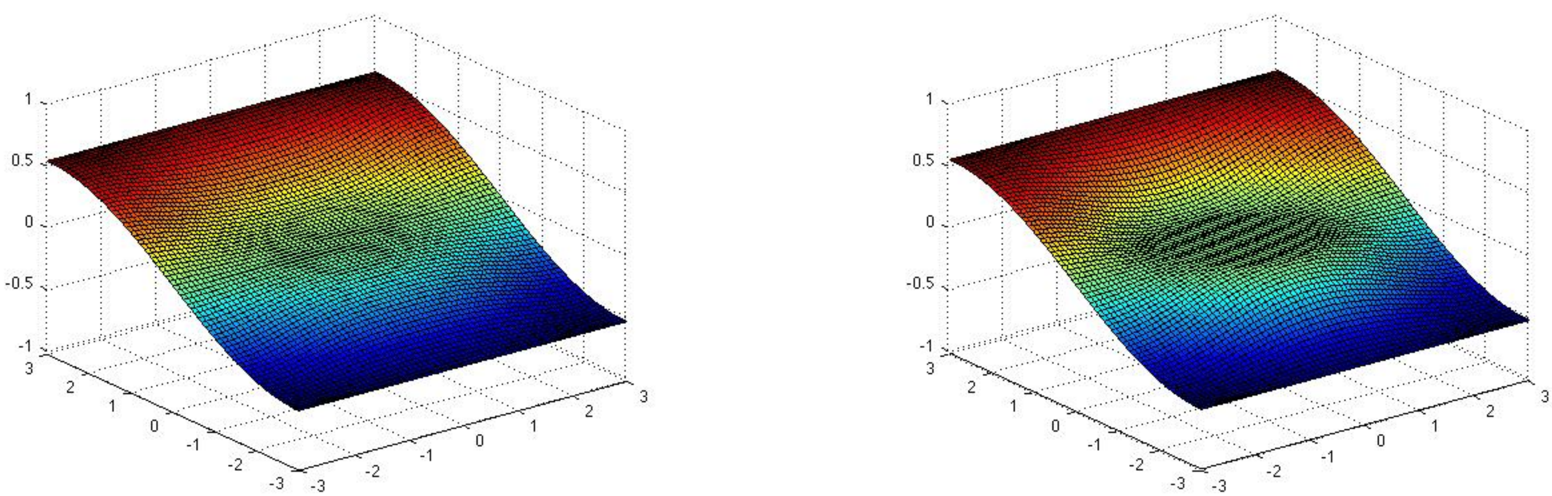




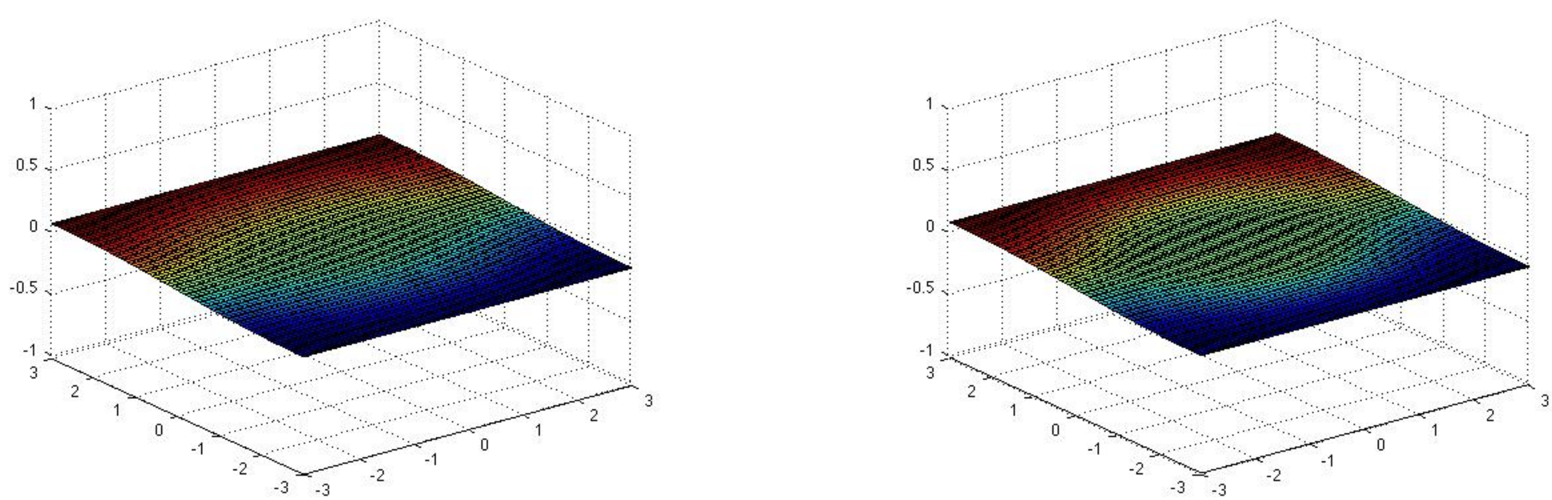
$t = 0$


1
0.5
0
-0.5
-1
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

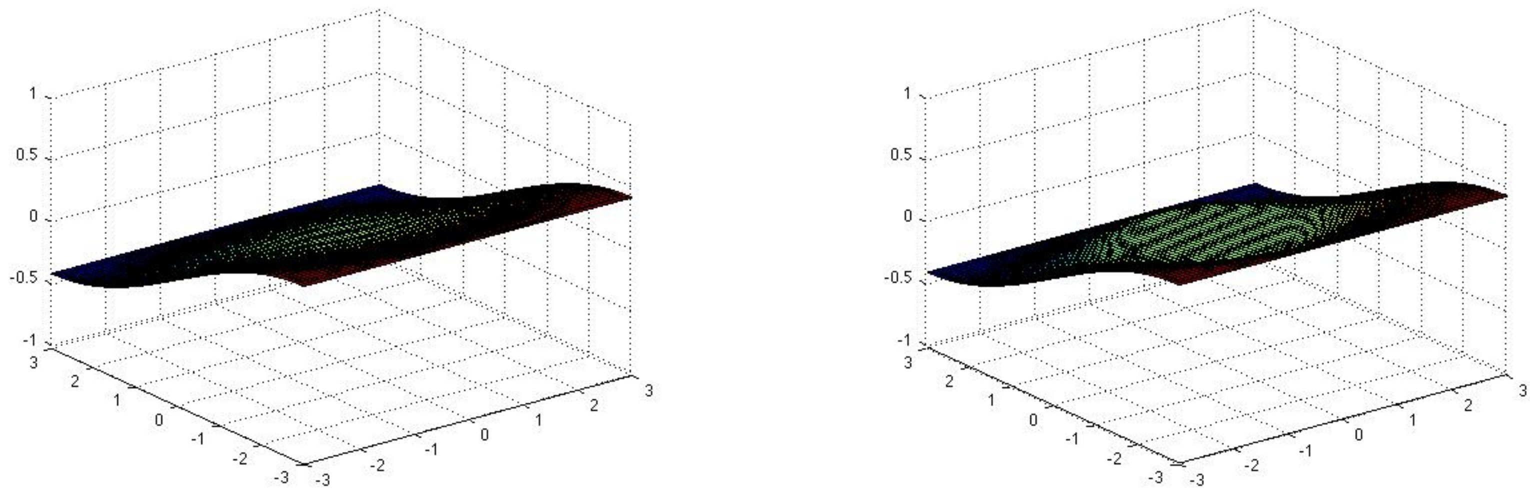
$t = .5$


1
0.5
0
-0.5
-1
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

$t = 1$


1
0.5
0
-0.5
-1
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

$t = 1.5$



$t = 2$

As before, the two solutions pictured above would be indistinguishable. An outside observer would have no way to determine whether or not the wave was being “bent” around a hidden object. This gives further evidence that cloaking is theoretically possible.

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