The Hamiltonicity of Subgroup Graphs

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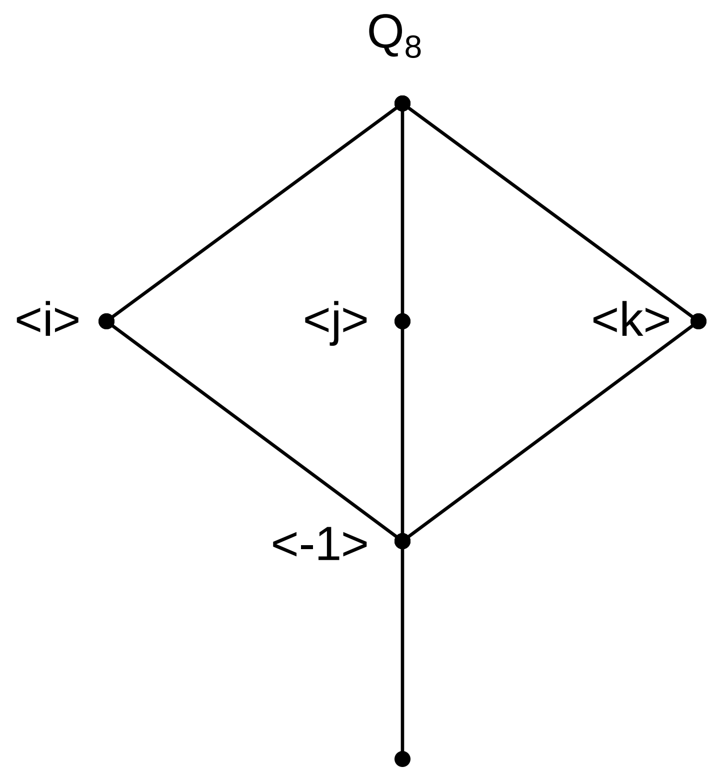
- A graph is a set of vertices, V, and a set of edges, E, (denoted by {v1,v2}) where v1,v2 $\in$ V and {v1,v2} $\in$ E if there is a line between v1 and v2.

- A subgroup graph of a group G is a graph where the set of vertices is all subgroups of G and the set of edges connects a subgroup to a supergroup if and only if there are no intermediary subgroups.

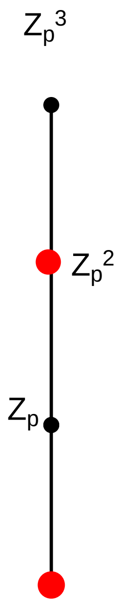
Examples of Subgroup Graphs

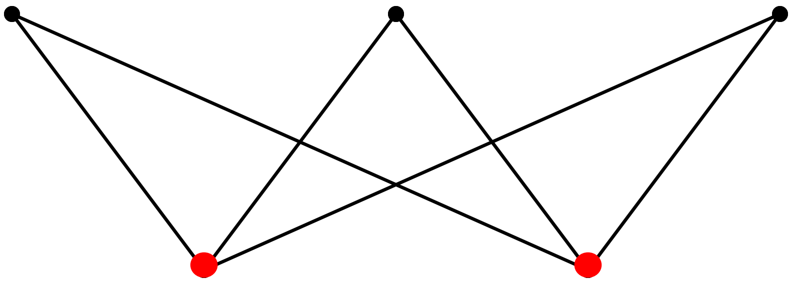


Definitions

- A graph is bipartite if the set of vertices V can be broken into two subset V_1 and V_2 where there are no edges connecting any two vertices of the same subset.

Examples of Bipartite Graphs



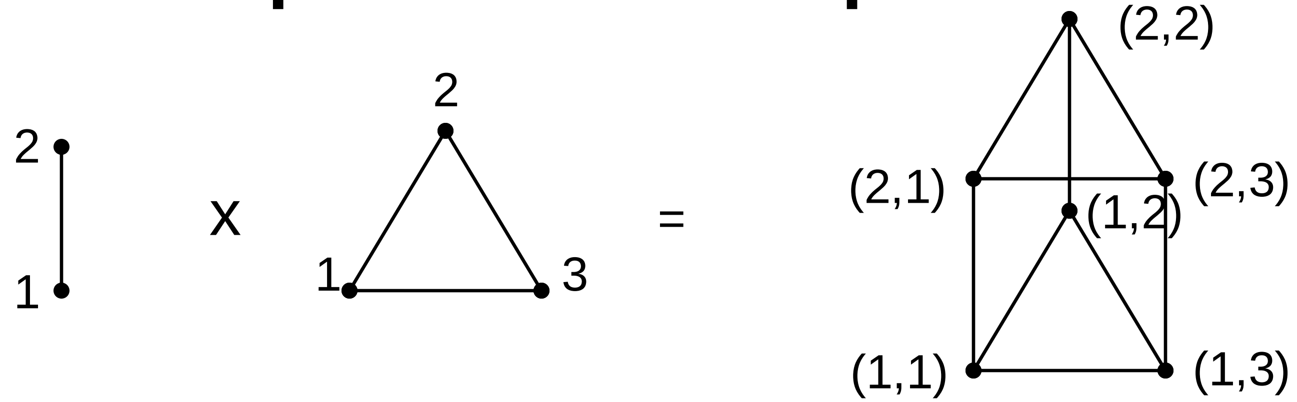


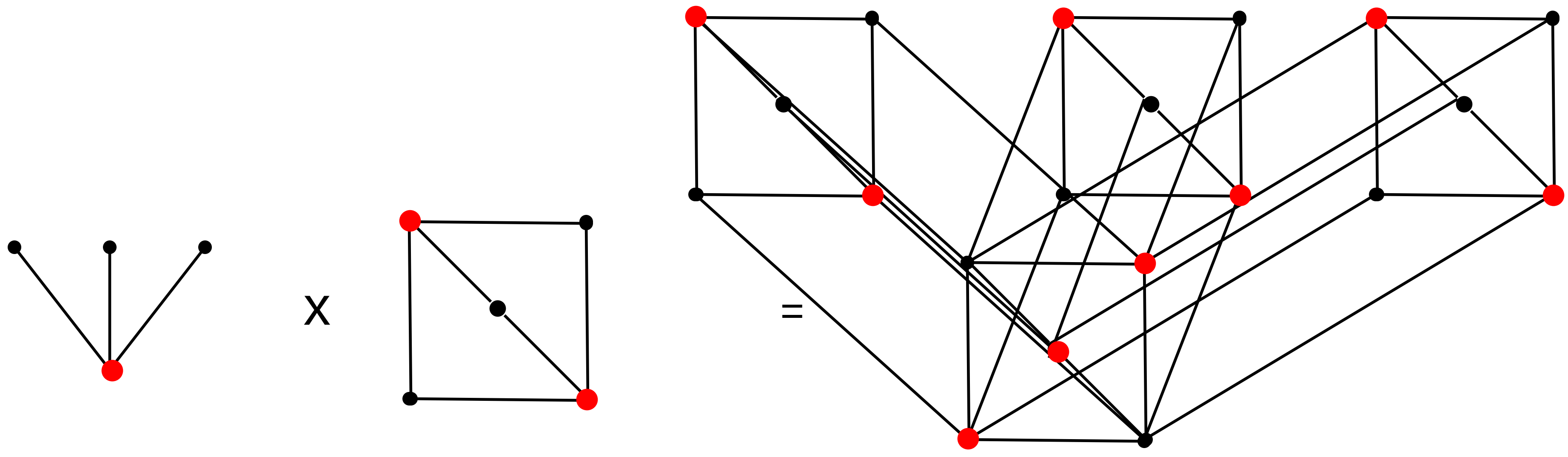
Graph Cartesian Products

- Let G and H be graphs, then the vertex set of G x H is V(G) x V(H).

- An edge, {(g,h),(g`,h`)}, is in the edge set of G x H if g = g` and h is adjacent to h` or h = h` and g is adjacent to g`.

Examples of Graph Product



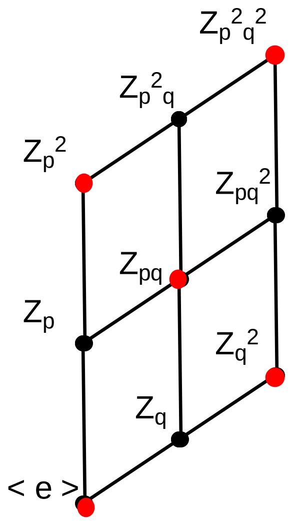


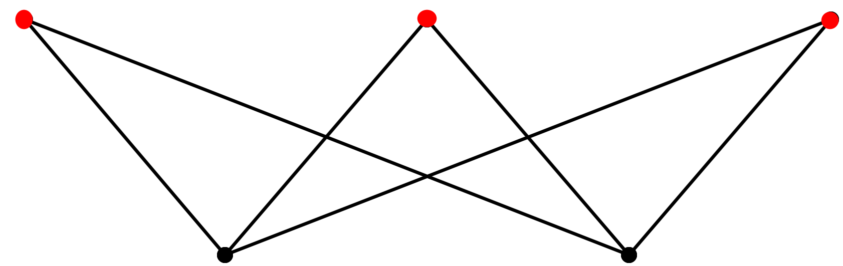
Results on Graph Products

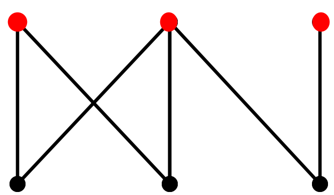
- The graph product of two bipartite graphs is bipartite.

- The difference in the size of the partitions of a graph product is the product of the difference in the size of the partitions of each graph in the product.

- Unbalanced bipartite graphs are never Hamiltonian. The reverse is not true in general.





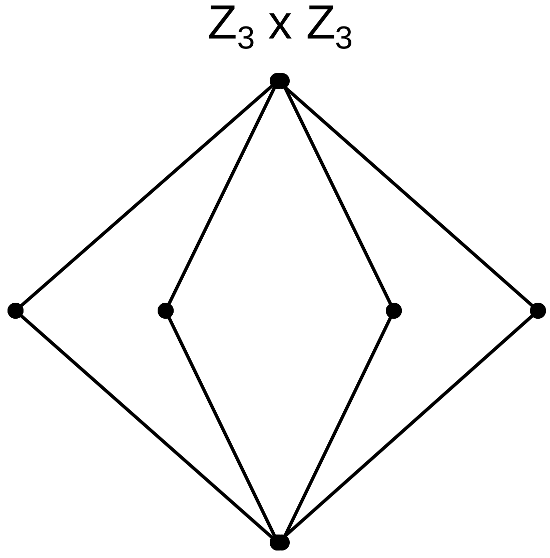


- For two relatively prime groups, G_1 and G_2, the subgroup graph of G_1 X G_2 is isomorphic to the graph cartesian product of the subgroup graphs of G_1 and G_2.

- The fundamental theorem of finite abelian groups says that every group can be represented as the cross product of cyclic p-groups.

Finite Abelian Groups

- Finite abelian p-groups are balanced if and only if $|G| = p^n$ where n is odd.



Finite Abelian Groups

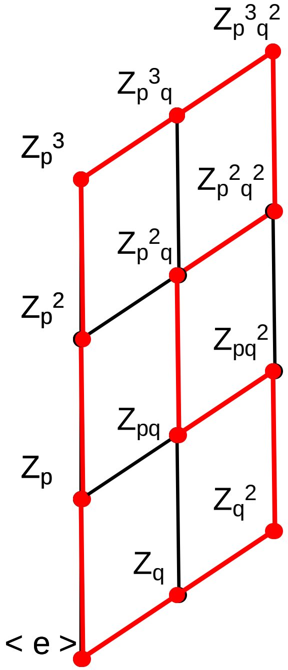
- A finite abelian group is balanced if and only if when decomposed into p-groups $G_{p_1^{\alpha_1}}$ x … x $G_{p_n^{\alpha_n}}$, α_i is odd for some i .

Cyclic Groups

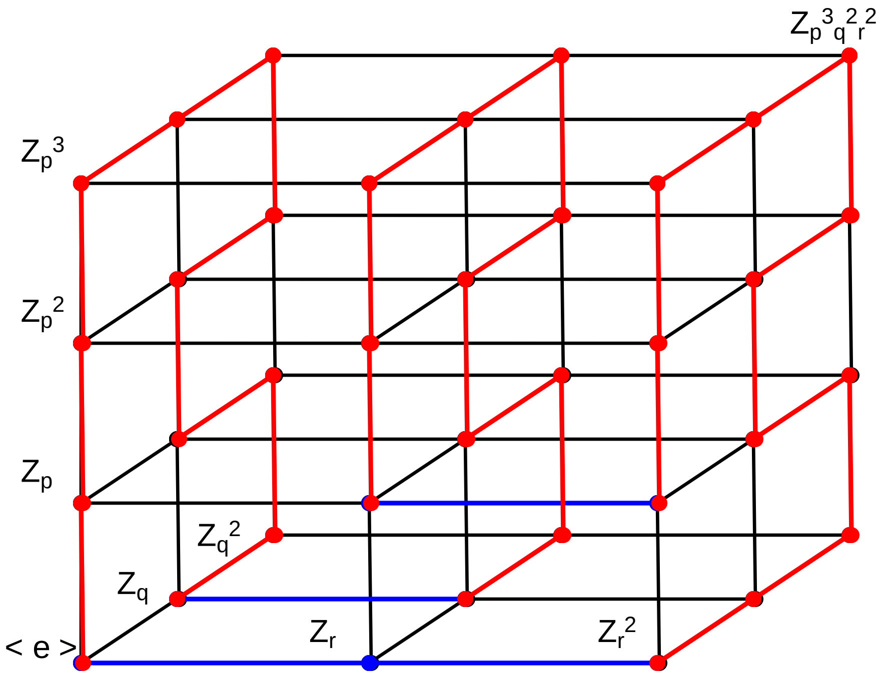
- Cyclic p-groups are nonhamiltonian.
- Cyclic groups, $\mathbf{Z}_{p_1^{\alpha_1} \cdots p_n^{\alpha_n}}$, with more than one prime factor are hamiltonian if and only if there is at least one α_i that is odd.

Cyclic Groups

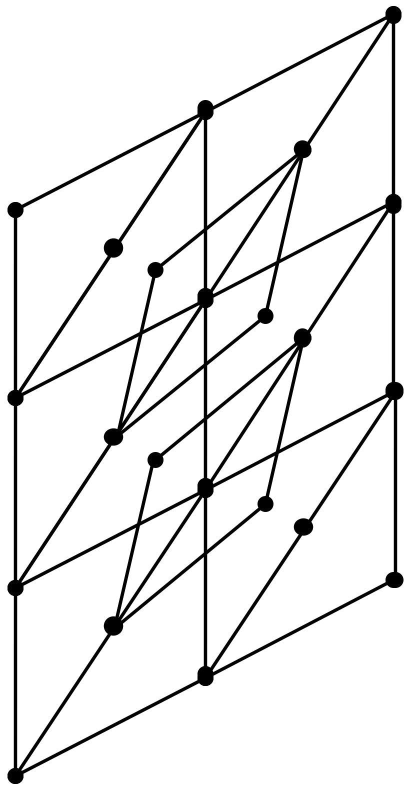
Cyclic Groups

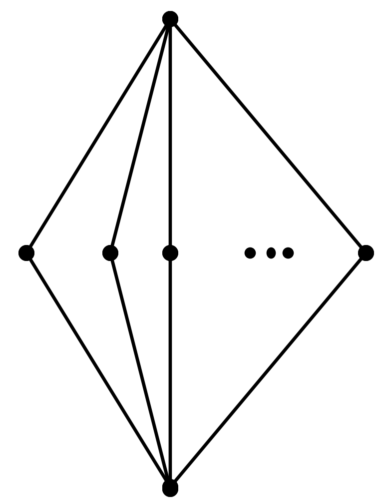


Cyclic Groups

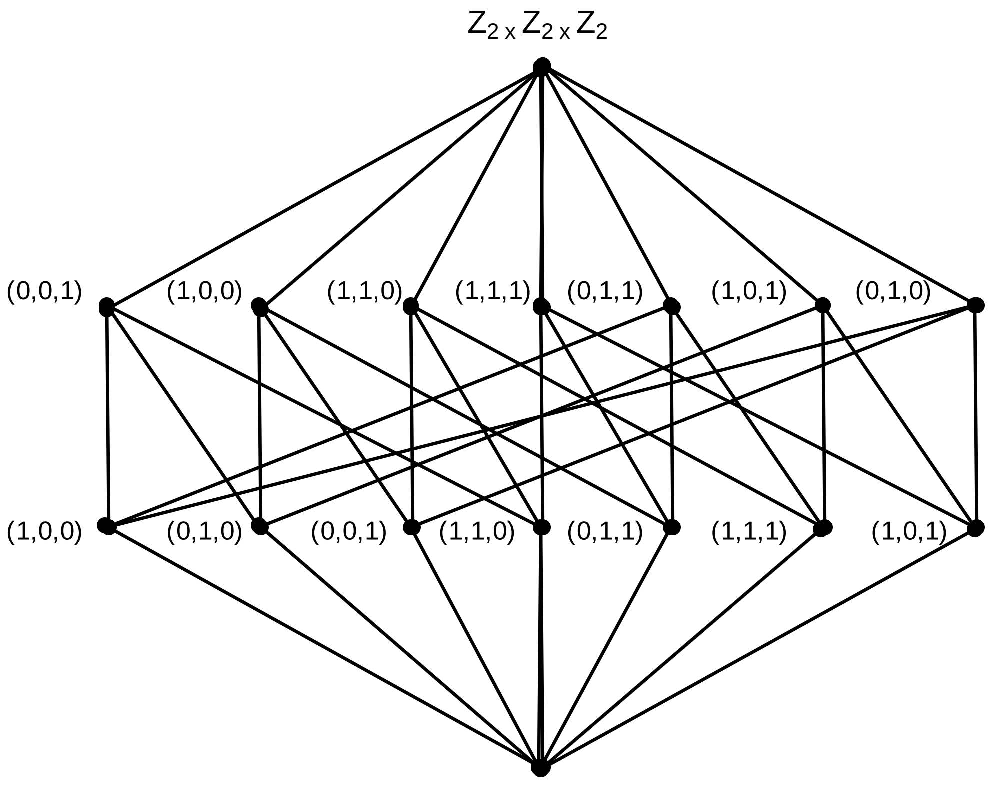


$\mathbf{Z}_{p^{\alpha}} \times \mathbf{Z}_{p^{\beta}}$ is nonhamiltonian.





$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$



$$\mathbf{Z}_p \times \mathbf{Z}_p \times \mathbf{Z}_p$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^1 \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^5 \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

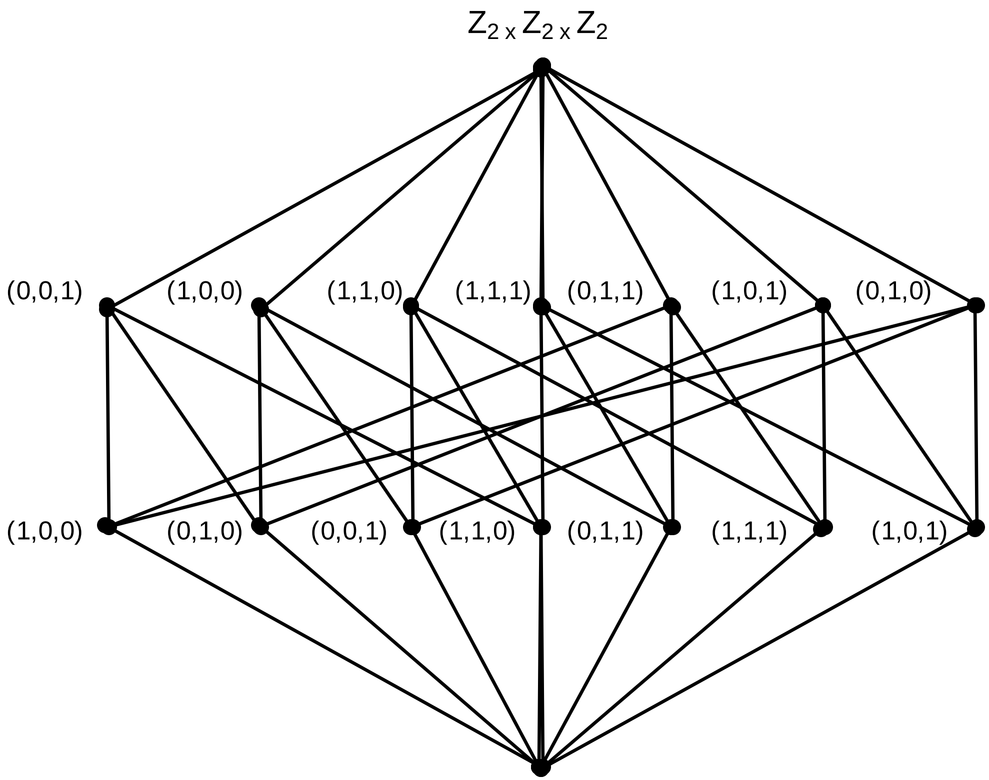
$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^6 \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^3 \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^7 \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^4 \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$



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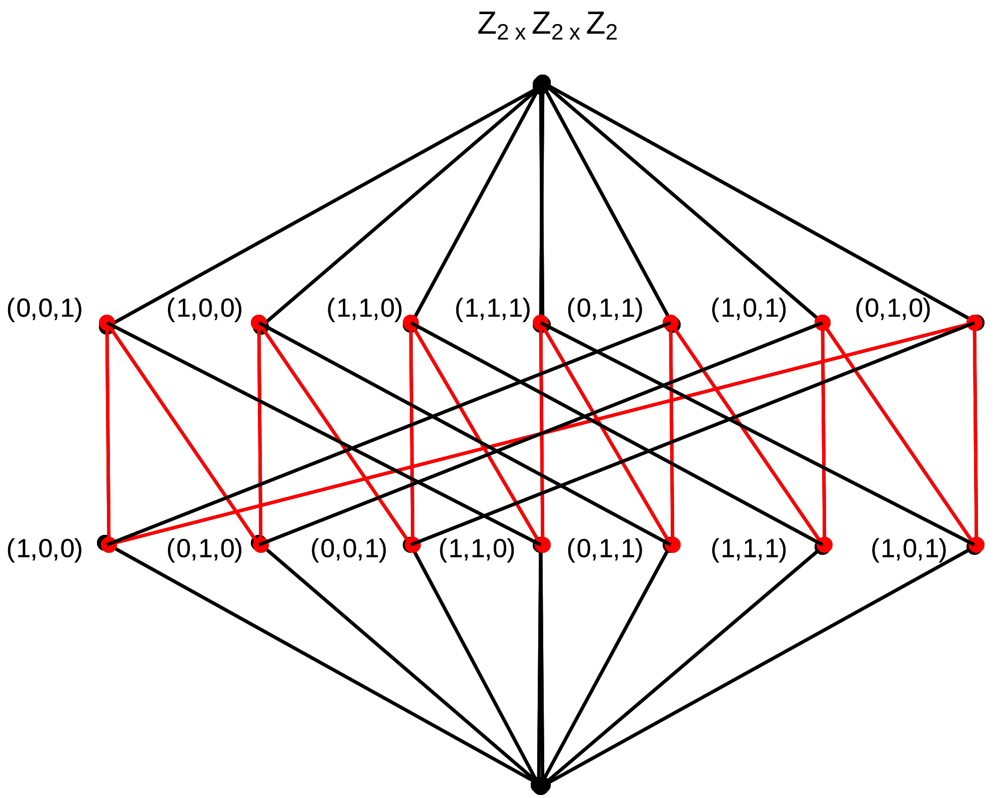

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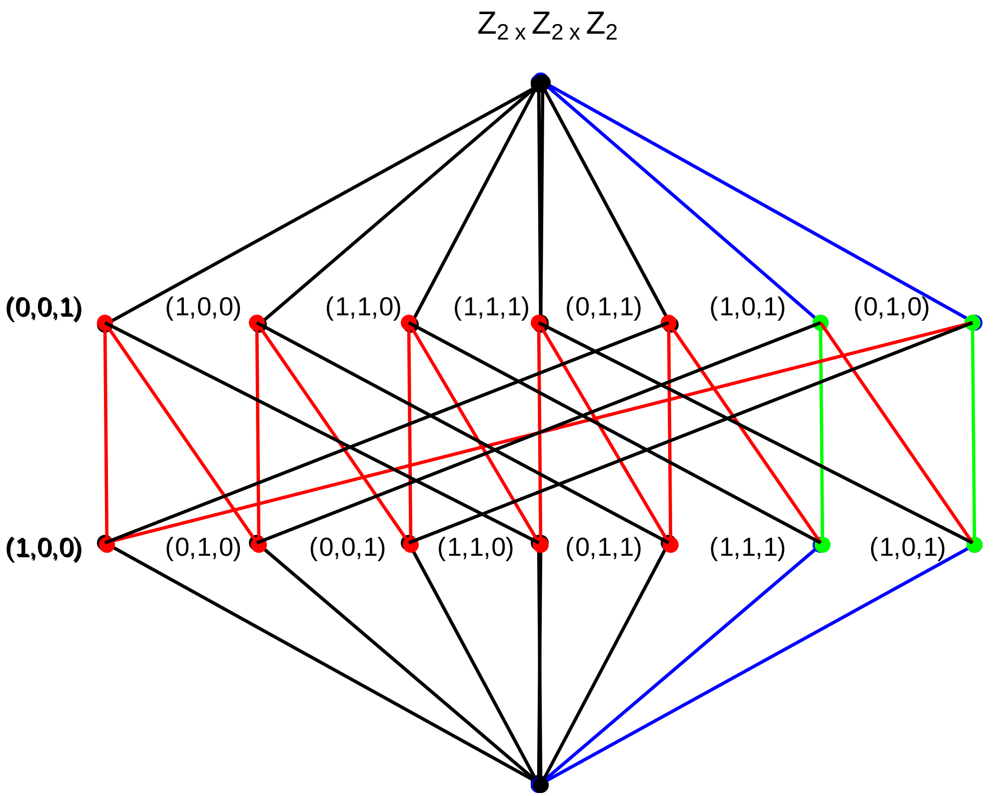

Dihedral Groups

- Dihedral groups are bipartite and the difference in the size of the partitions of $D_{2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}}$ is $\prod_{i=1}^{k}\frac{p_i^{\alpha_i+1}-(-1)^{\alpha_i+1}}{p_i+1}+\theta(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n})$, where

$$\theta(x)=\begin{cases}-1 & \text{if x is square}\\ 0 & \text{otherwise}\end{cases}$$

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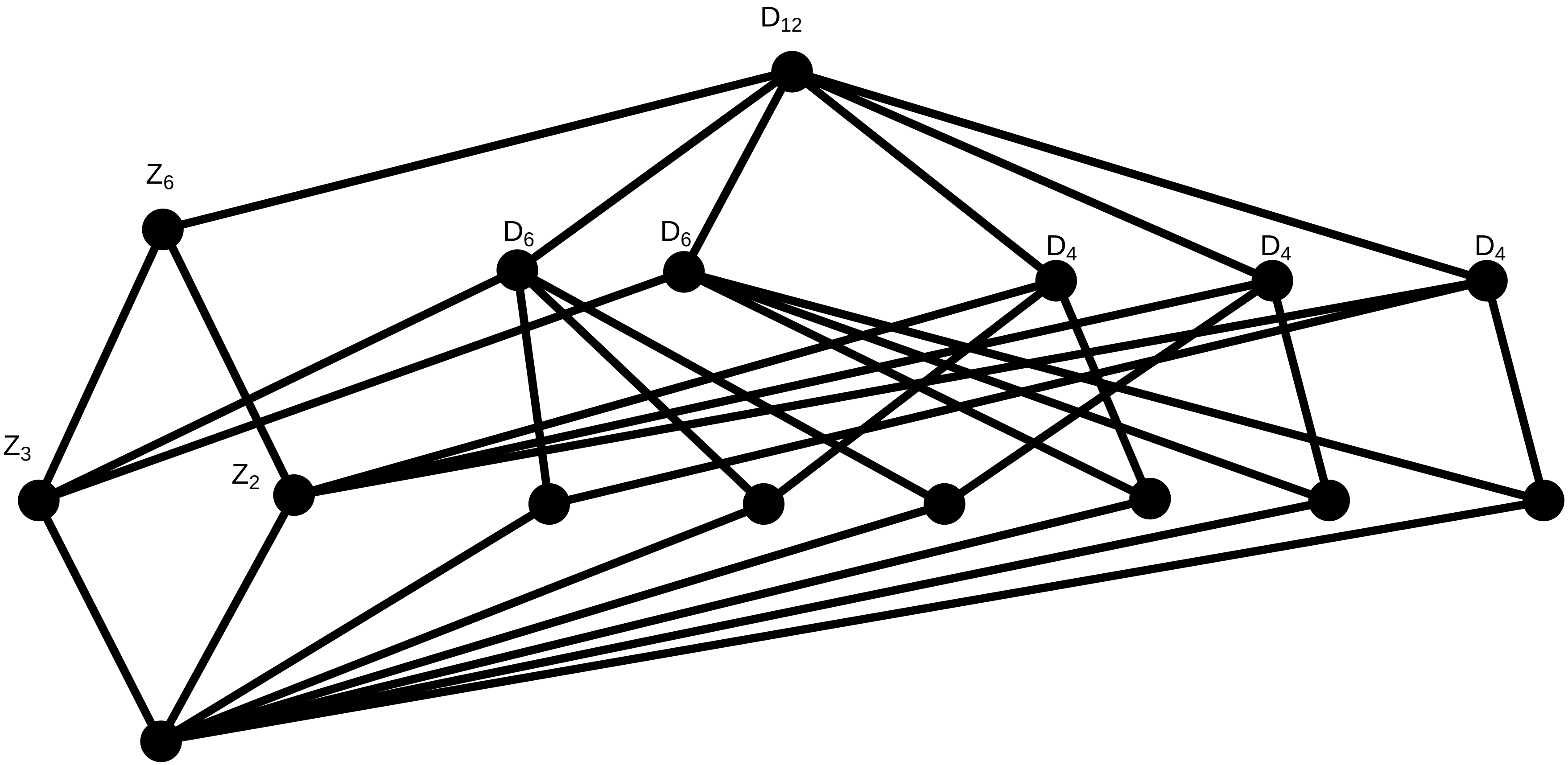

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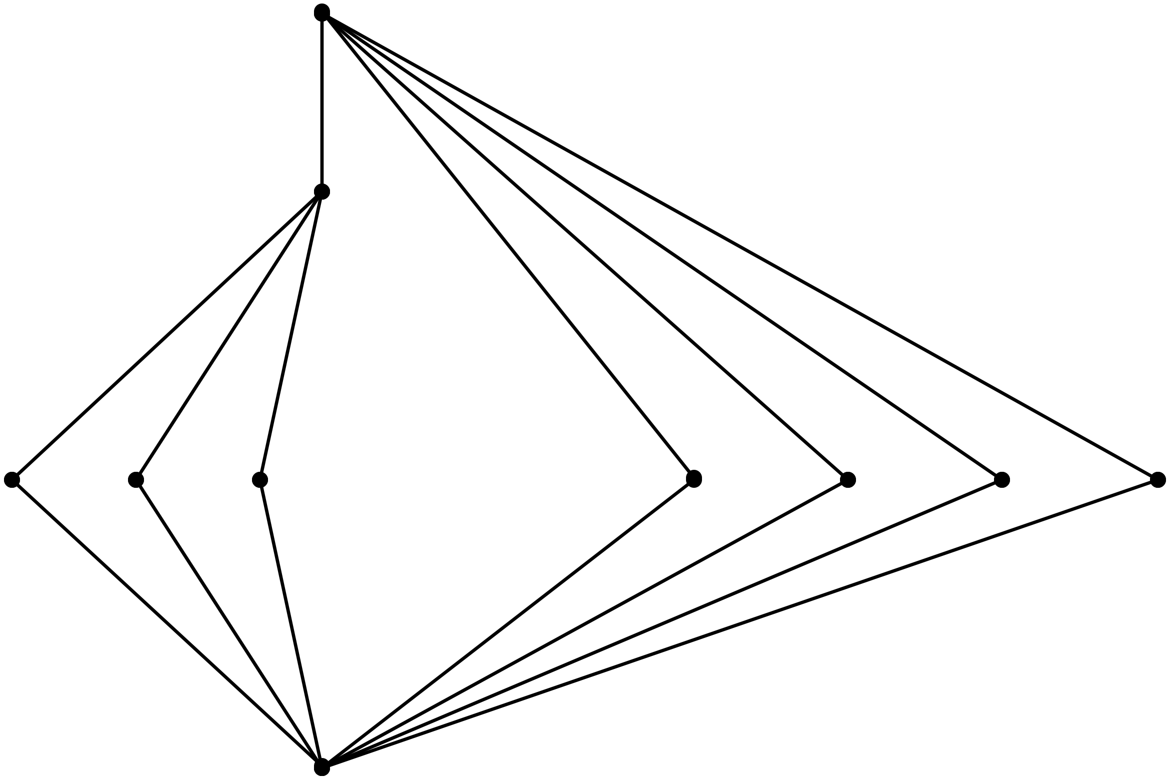

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