On the Non-Orientable Genus of Zero Divisor Graphs

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Aug. 1, 2007

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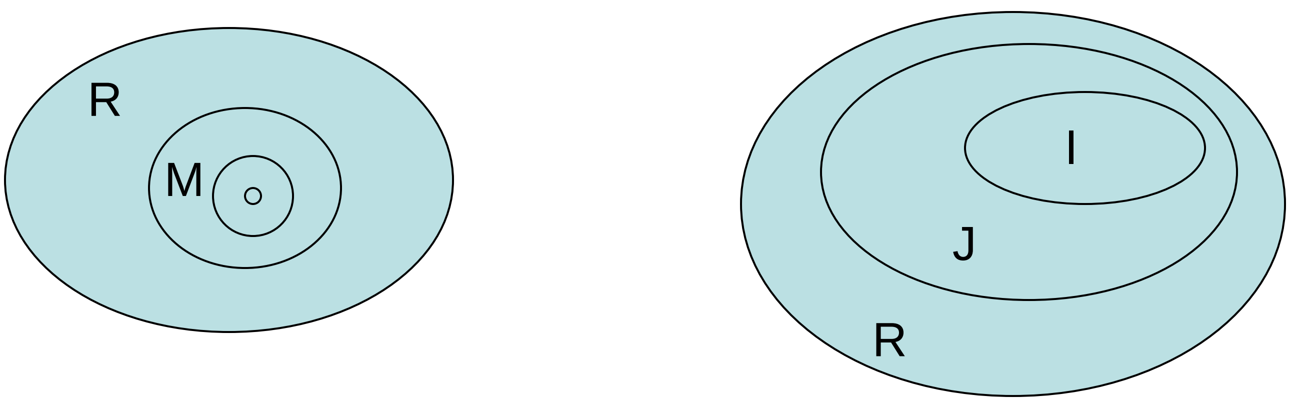
Local Rings

Definition: A maximal ideal of a ring R is an ideal M, not equal to R, such that there are no ideals “in between” M and R.

Definition: A finite commutative ring R is local if it has a unique maximal ideal.



M is the maximal ideal of R.

I is not the maximal ideal of R

Zero Divisors

Definition: An element $a \in R$

is a zero divisor of R if there is an element $b \in R$

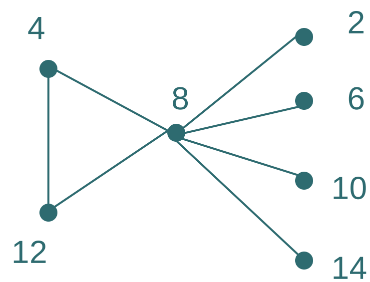
such that $ab = 0.$

When R is a local ring, the maximal ideal is exactly the set of zero divisors.

Zero Divisor Graph

Definition: The zero divisor graph of R, denoted $\Gamma(R)$ is the graph whose vertex set is the set of zero divisors of R and whose edge set is $E = \{\{a,b\} \subseteq Z(R) \mid ab = 0\}$.

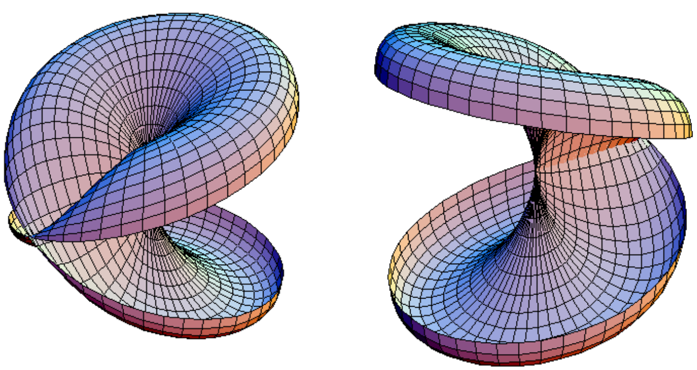
Example: $\Gamma(\mathbf{Z}_{16}) =$



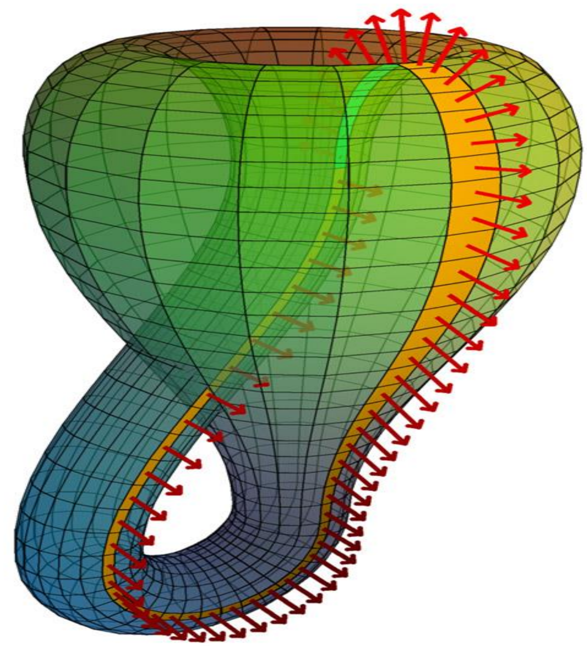
$$Z(\mathbf{Z}_{16}) = \{0, 2, 4, 6, 8, 10, 12, 14\}$$

Non-Orientable Surfaces of Genus 1 and 2

A non-orientable surface cannot be embedded in 3-dimensional space without intersecting itself.



Real Projective Plane



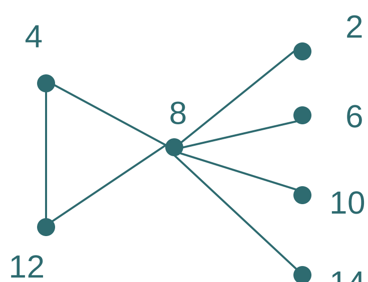
Klein Bottle

Genus

Definition: The non-orientable genus of a zero divisor graph is the smallest integer k such that the graph can be drawn on a surface of genus k without edges crossing.

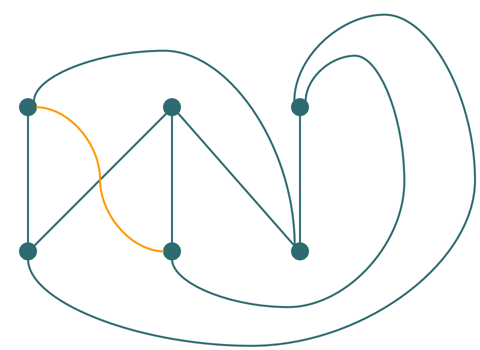
Example: A planar graph has genus 0.

$$\Gamma(\mathbf{Z}_{16}) =$$

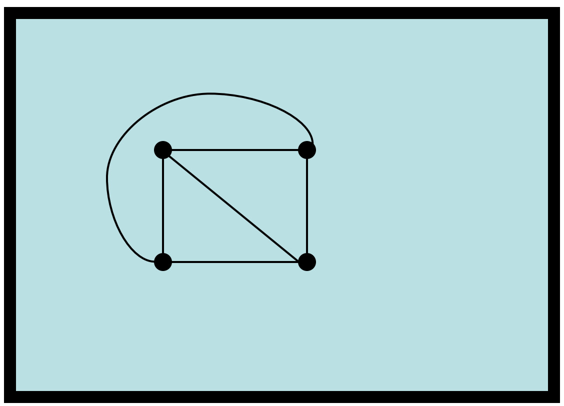


Example: $K_{3,3}$

Cannot be drawn on the plane without intersecting itself.

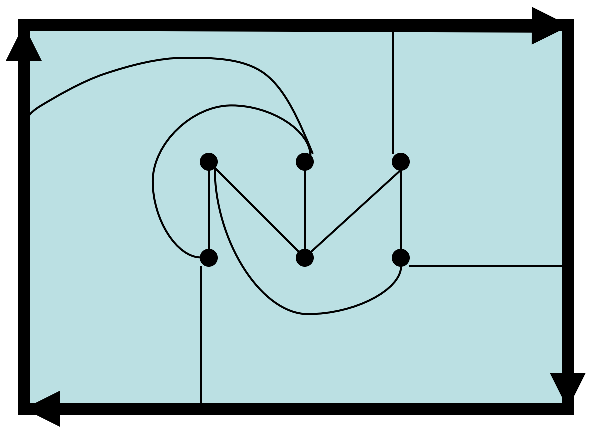


Fundamental Polygons



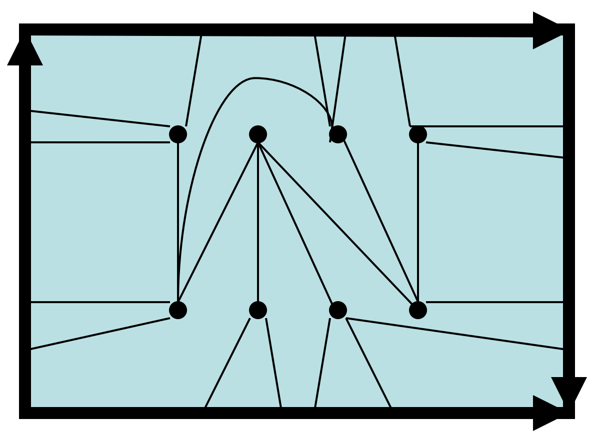
K_4

Planar



$K_{3,3}$

Genus 1



$K_{4,4}$

Genus 2

Formulas for Finding the Genus of a Graph

Formulas for determining the non-orientable genus of complete graphs and complete bipartite graphs:

$$\overline{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$$ for $n \geq 3$ with the exception :

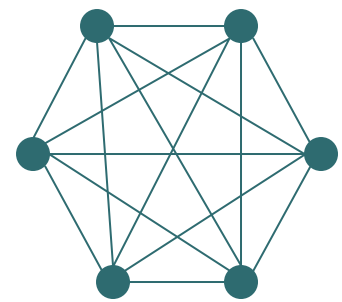
$$\overline{\gamma}(K_7) = 3$$

$$\overline{\gamma}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil$$ for $m, n \geq 2.$

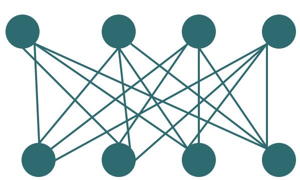
The Genus of Complete Graphs

Example: Complete graphs on n vertices:

$$\bar{\gamma}(K_n) = 0 \quad for \quad n = 1, 2, 3, 4$$
$$\bar{\gamma}(K_n) = 1 \quad for \quad n = 5, 6$$
$$\bar{\gamma}(K_n) > 2 \quad for \quad n \geq 7.$$



Genus 1



Genus 2

Example: Complete bipartite graphs:

$$\gamma(K_{m,n}) = 0 \quad for \quad m = 1, 2 \quad and \;\; for\ all\ n.$$
$$\gamma(K_{3,3}) = \gamma(K_{3,4}) = 1$$
$$\gamma(K_{4,4}) = \gamma(K_{3,5}) = \gamma(K_{3,6}) = 2.$$

My Game Plan:

Theorem: Every finite commutative ring can be written as the product of local rings.

Local Rings of order:

p

p^2

p^3

p^4

p^5

p is prime

Non-local Rings

With two local factors:

$F_p \times F_q$

$F_p \times F_{q^2}$

$F_p \times F_{q^3}$

$F_p \times \mathbf{Z}_{q^2}$

$F_p \times \mathbf{Z}_{q^3}$

With three local factors:

$F_p \times F_p \times F_p$

$F_p \times F_q \times F_r$

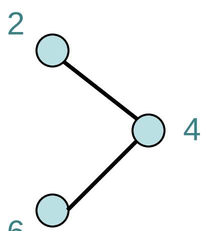
$F_p \times F_q \times F_{r^2}$

$F_p \times F_q \times \mathbf{Z}_{r^2}$

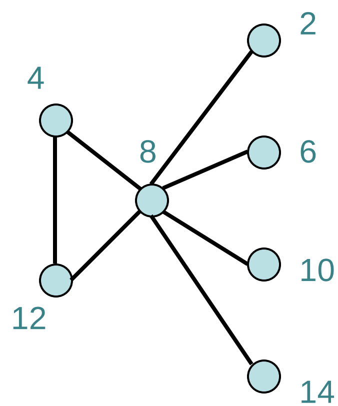
With four local factors: $F_p \times F_p \times F_p \times F_p$

Examples of Local Rings when p=2

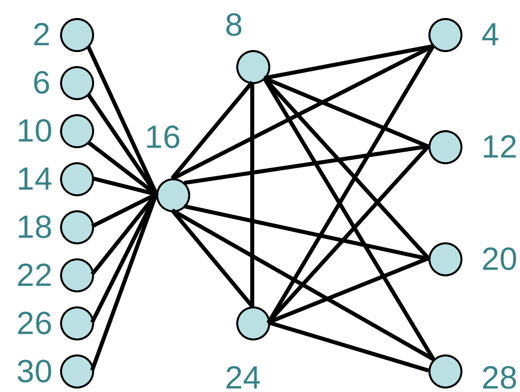
$\mathbf{Z}_8$



$\mathbf{Z}_{16}$



$\mathbf{Z}_{32}$



$\mathbf{Z}_8$ has p^3 elements.	$\mathbf{Z}_{16}$ has p^4 elements.	$\mathbf{Z}_{32}$ has p^5 elements.

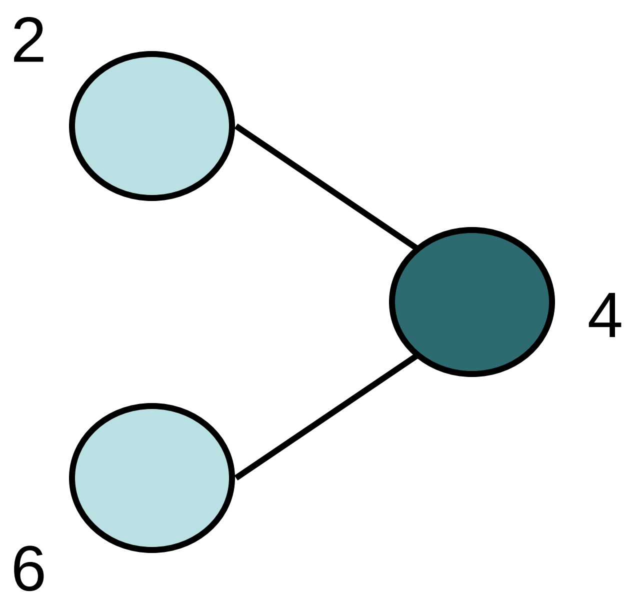
The Maximal Ideal and the Zero Divisor Graph

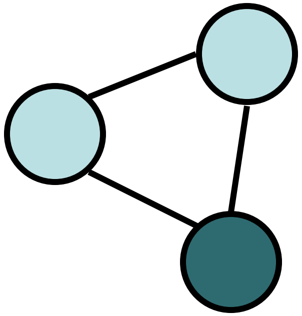
$\mathbf{Z}_8$

M = {zero divisors} = (2) = {0, 2, 4, 6}

M^2 = (4) = {0, 4}

M^3 = (0) = {0}





The Maximal Ideal and the Zero Divisor Graph

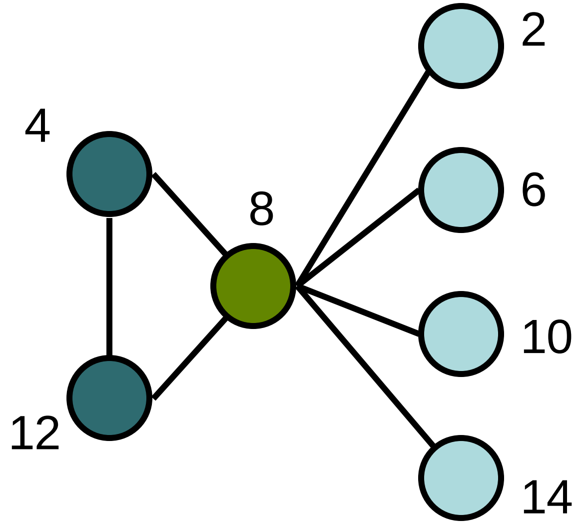
$\mathbf{Z}_{16}$

$M = (2) = \{0, 2,4,6,8,10,12,14\}$

$M^2 = (4) = \{0, 4, 8, 12\}$

$M^3 = (8) = \{0, 8\}$

$M^4 = (0) = \{0\}$



The Maximal Ideal and the Zero Divisor Graph

$\mathbf{Z}_{32}$

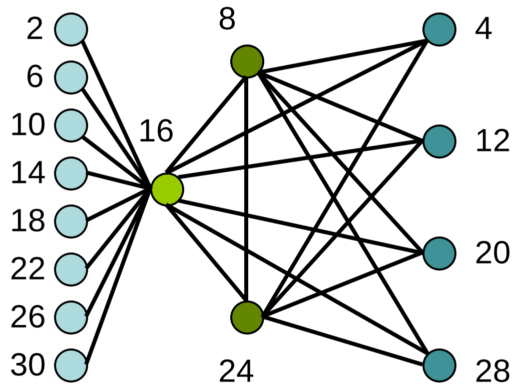
$M = (2) = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30\}$

$M^2 = (4) = \{0, 4, 8, 12, 16, 20, 24, 28\}$

$M^3 = (8) = \{0, 8, 16, 24\}$

$M^4 = (16) = \{0, 16\}$

$M^5 = (0) = \{0\}$

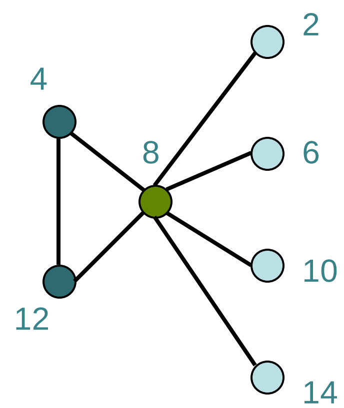


Collapsing the Graphs

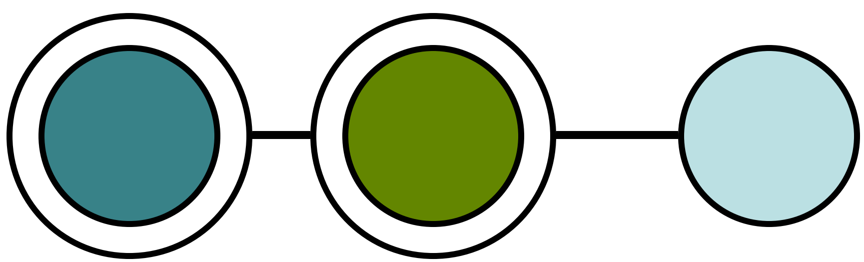
$M - M^2 = (2) - (4) = \{2, 6, 10, 14\}$

$M^2 - M^3 = (4) – (8) = \{4, 12\}$

$M^3 - M^4 = (8) – (16) = \{8\}$



$\mathbf{Z}_{16}$



Making Vertex Sets From Equivalence Relations

Definition: The set of annihilators of a ring element a is

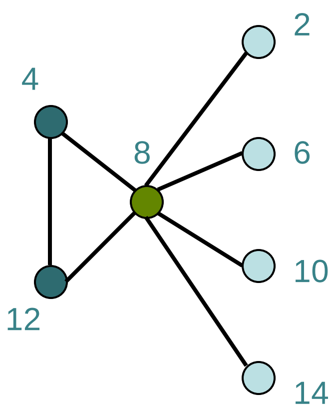
$$ann(a) = \{r \in R \mid ra = 0\}.$$

Equivalence Relation: $a : b \ \ if \ \ ann(a) = ann(b)$

$$[a] = \{b \in R \mid a : b\}$$

In other words, two ring elements *a* and *b* are equivalent if they have the same annihilators.

An Example of an Equivalence Relation



$\mathbf{Z}_{16}$

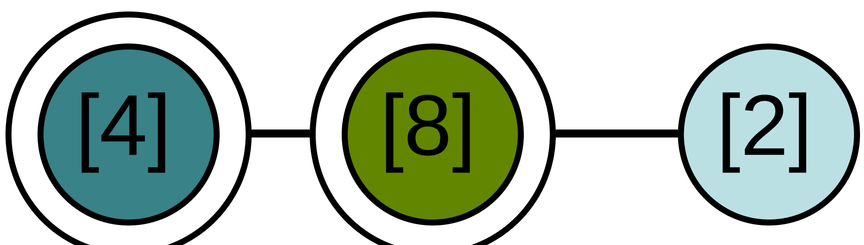
$$[2] = \{ b \in R \,|\, 2 : \;\; b \} = \{2, 6, 10, 14\}$$

$ann(2) =$

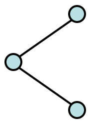
$ann(6) =$

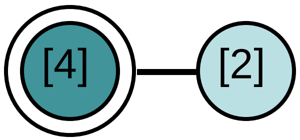
$ann(10) =$

$ann(14) = \{0, 8\}$

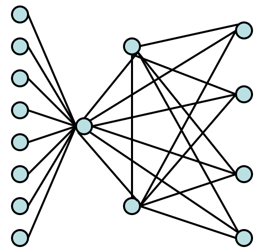


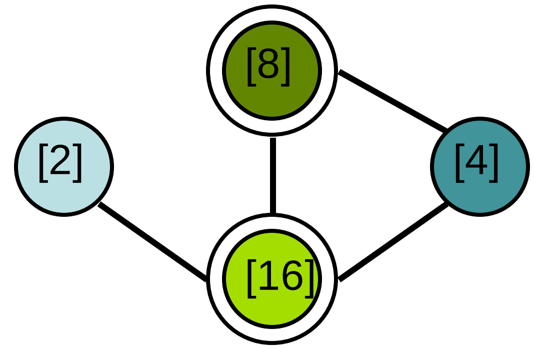
Collapsing the Graphs of Integer Rings



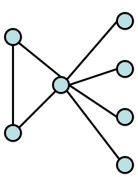


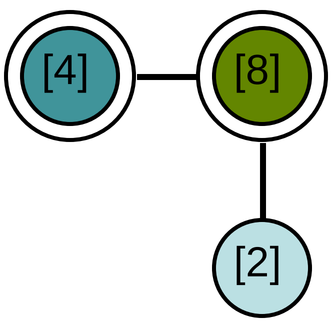
$\mathbf{Z}_{8}$



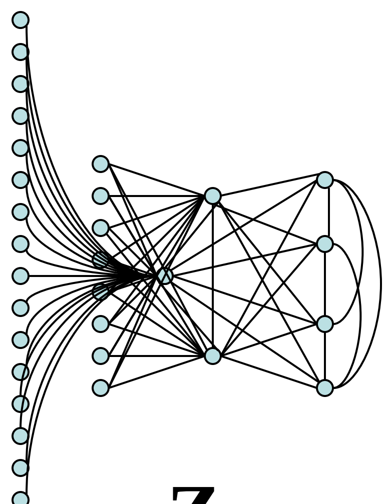


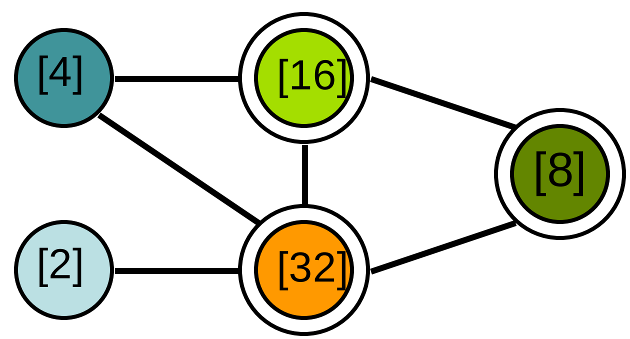
$\mathbf{Z}_{32}$





$\mathbf{Z}_{16}$





$\mathbf{Z}_{64}$

What’s Next:

So far, we have been considering integer rings where M, M^2, M^3, …are each generated by one ring element.

$$M = (a)$$

$$M^2 = (a^2)$$

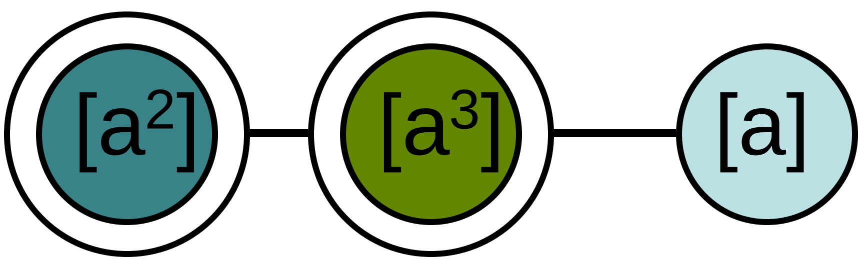
$$M^3 = (a^3)$$

.

.

.

$$M^n = \{0\}$$

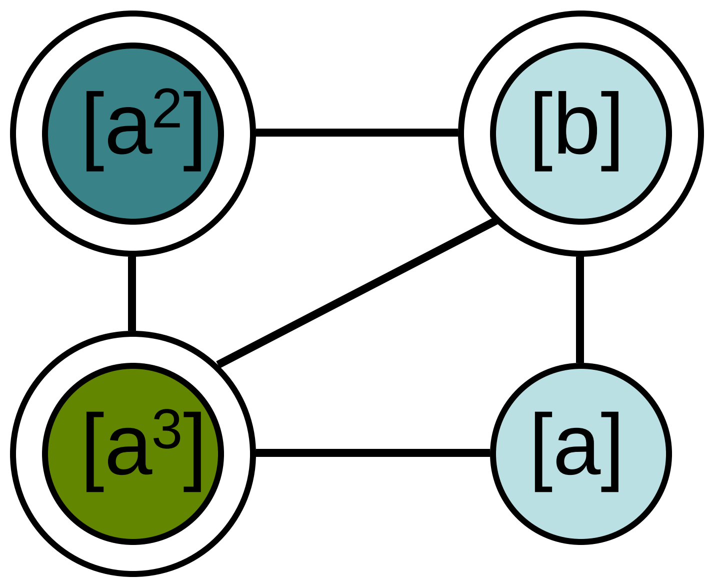


What’s Next:

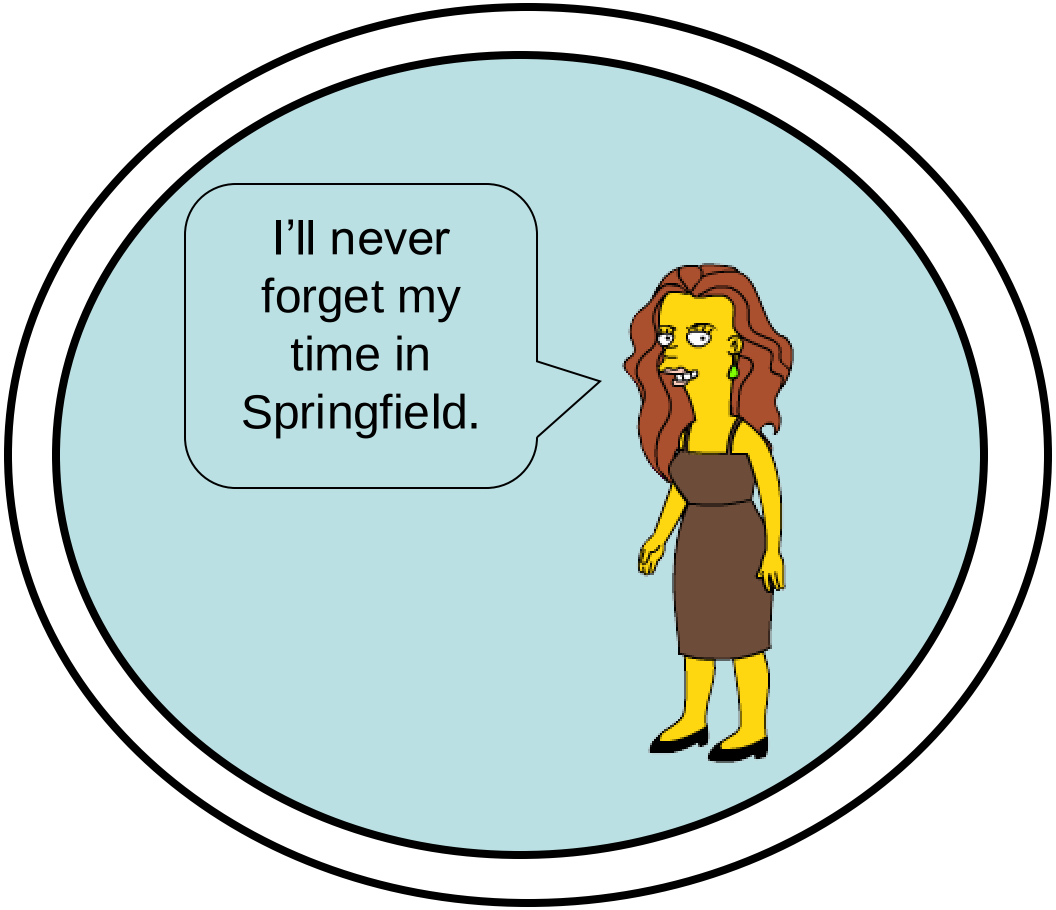
What happens when M is generated by more than one element? For example:

$$R = \mathbf{Z}_4[X]/(2X, X^4)$$
$$M = (2, X)$$
$$M^2 = (X^2)$$
$$M^3 = (X^3)$$
$$M^4 = \{0\}$$

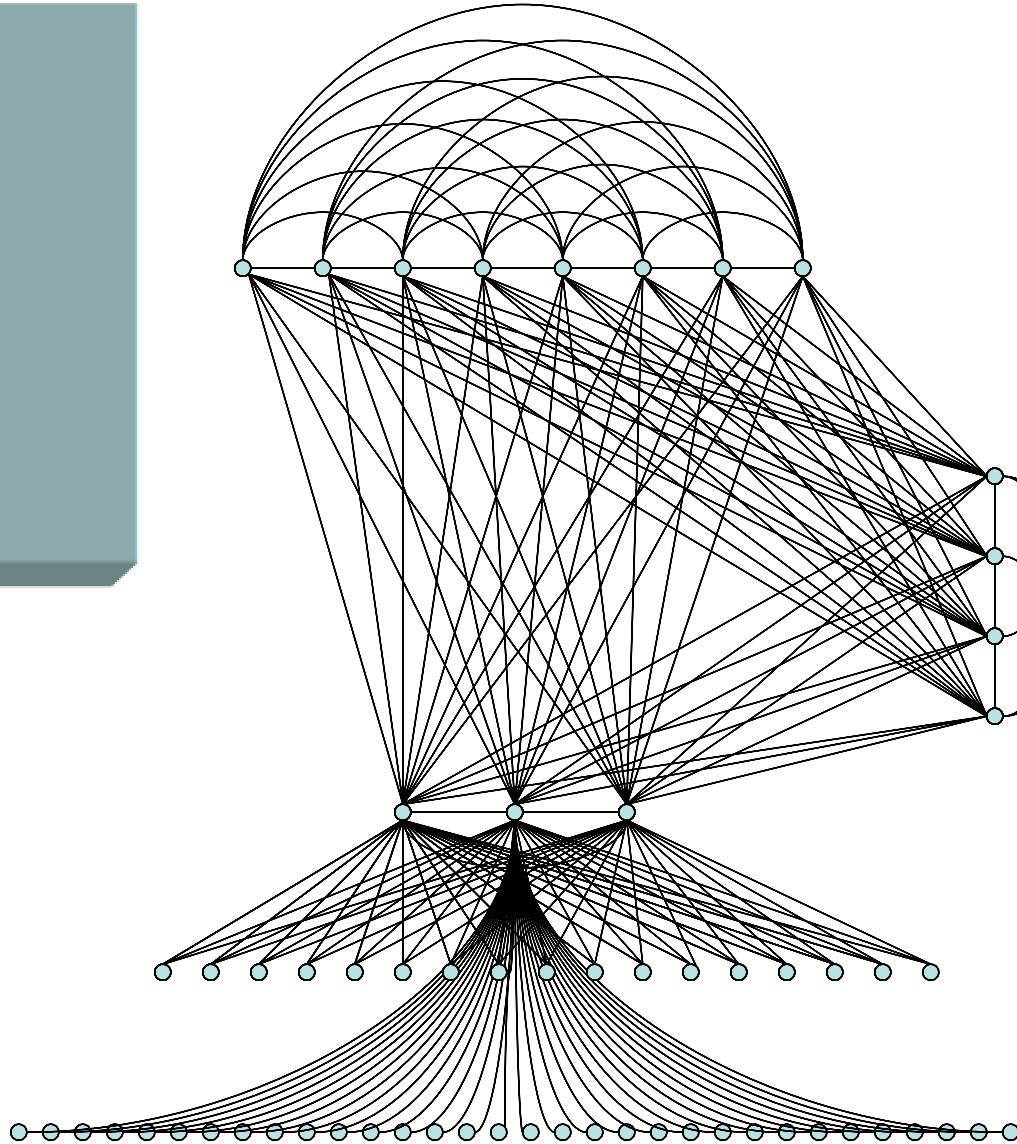
$$M = (a, b)$$
$$M^2 = (a^2)$$
$$M^3 = (a^3)$$
$$M^4 = \{0\}$$

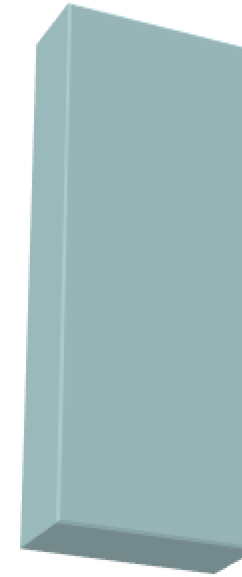


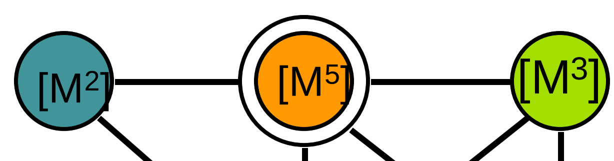
The End

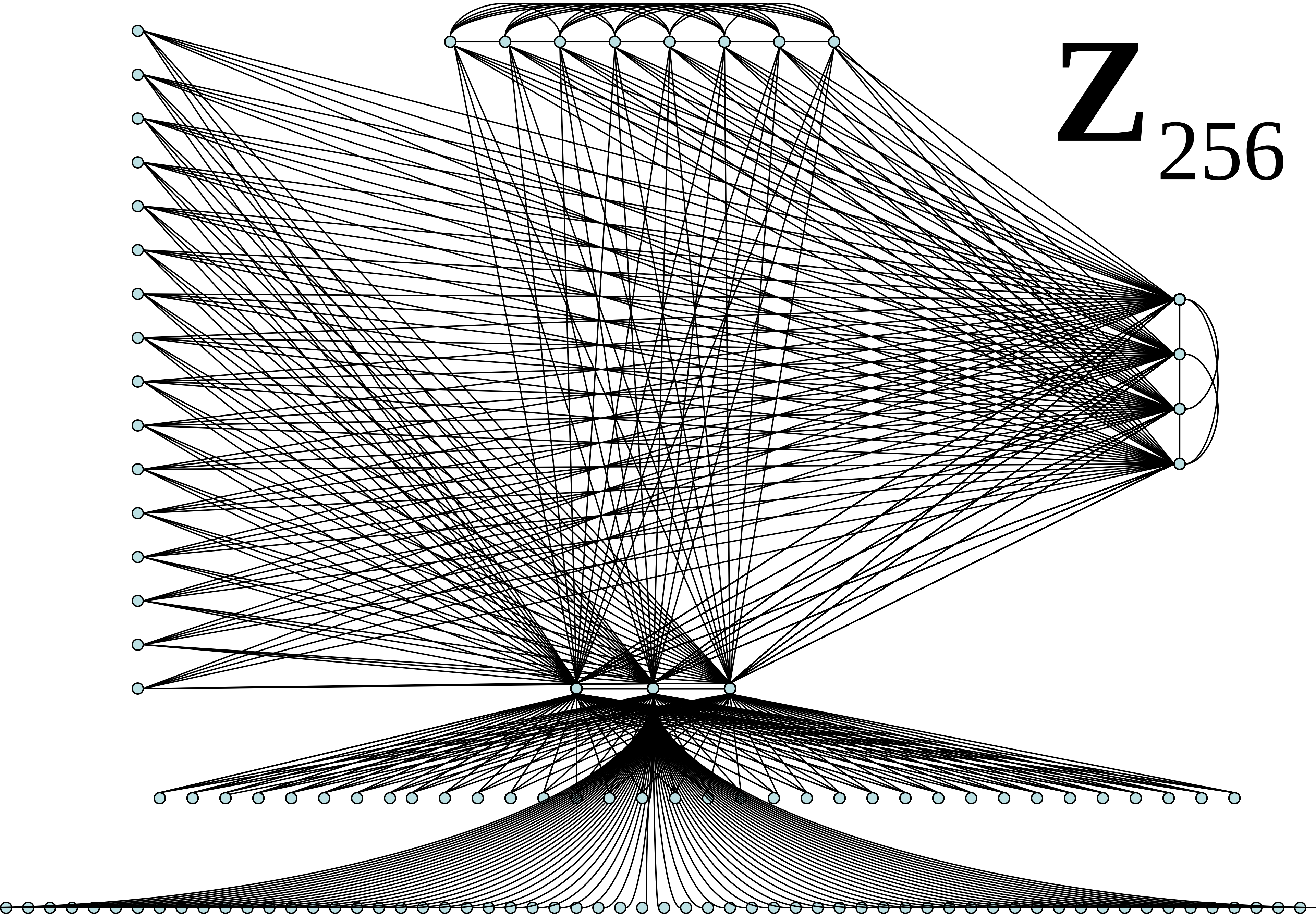


$\mathbf{Z}_{128}$

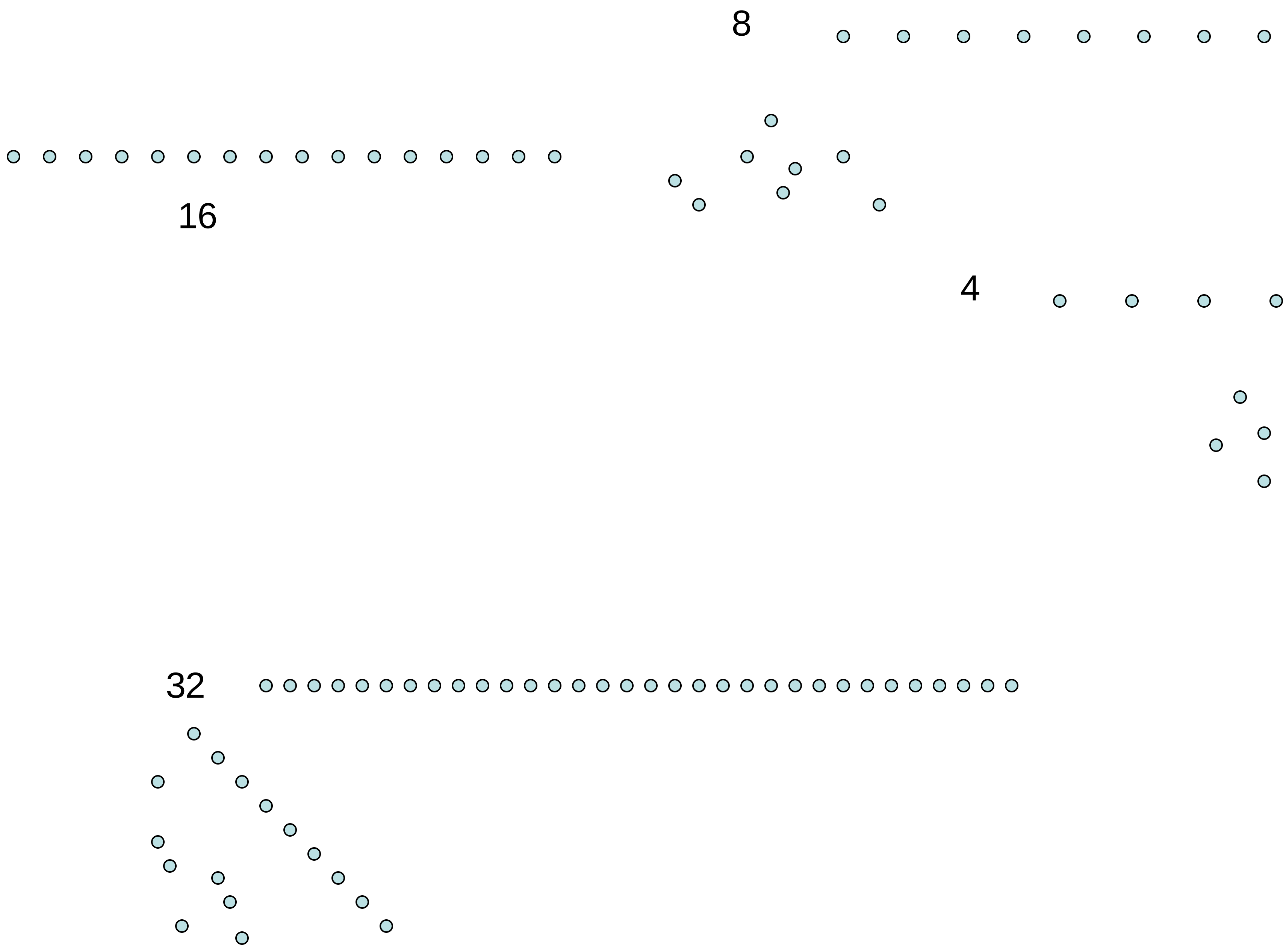








$\mathbf{Z}_{256}$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16
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32