Cohen-Macaulay Monomial Rings

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What's A Monomial Ring?

A monomial f is a polynomial with one term, i.e. a polynomial that may be expressed as:

$$f = x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$$

g_1, g_2, and g_3 are all monomials: $g_1 = x^5 y^2 z^9 \quad g_2 = y \quad g_3 = s^4 t^5$

h_1 and h_2 are not: $h_1 = x^2 y^2 + 1 \quad h_2 = y + z - x^5$

A monomial ring $R[f_1, \ldots, f_k]$ consists of all R-linear combinations of products of the monomials $f_1, \ldots, f_k$ where R is a ring.

We multiply monomials in the natural way:

$$r^4 s^5 \cdot r^2 s t^7 = r^6 s^6 t^7$$

If we identify the monomial $r^a s^b t^c$ with the vector (a, b, c), then monomial multiplication corresponds with vector addition:

$$r^4 s^5 \cdot r^2 s t^7 = r^6 s^6 t^7 \quad \Leftrightarrow \quad (4, 5, 0) + (2, 1, 7) = (6, 6, 7)$$

So corresponding to a monomial ring $R[f_1,\ldots,f_k]$ is the *semigroup*** *S* which is the collection of all non-negative linear combinations of the vectors identified with the exponents of $f_1,\ldots,f_k$ (this implies that we include the zero vector in the semigroup).

For example: $$R[s^2, st, t^2]$$

Corresponds with:

$$S = \langle (2,0),(1,1),(0,2) \rangle$$
$$= \{a(2,0)+b(1,1)+c(0,2) : a,b,c \in Z_{\geq 0}\}$$

The tuples in pointy brackets are called the *generators* of *S*. From now on I will talk about a monomial ring and its corresponding semigroup interchangeably.

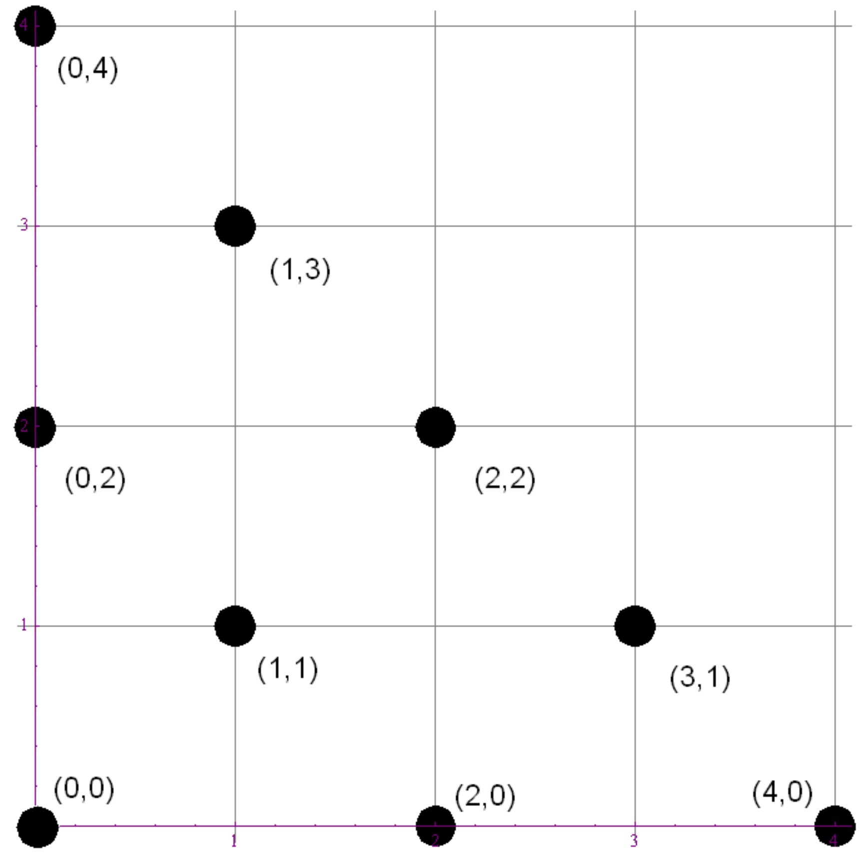
***** A group-like object which might not have inverse elements

Semigroup Lattice

- If the elements of our semigroup are either pairs or triplets, we may visualize the structure of a semigroup and its corresponding monomial ring with a semigroup lattice.

- Simply plot the elements of the semigroup in the plane or in 3-space.

- Here is part of the semigroup lattice for *S*=<(2,0),(1,1),(0,2)>



Simplicial Homogeneous Semigroups

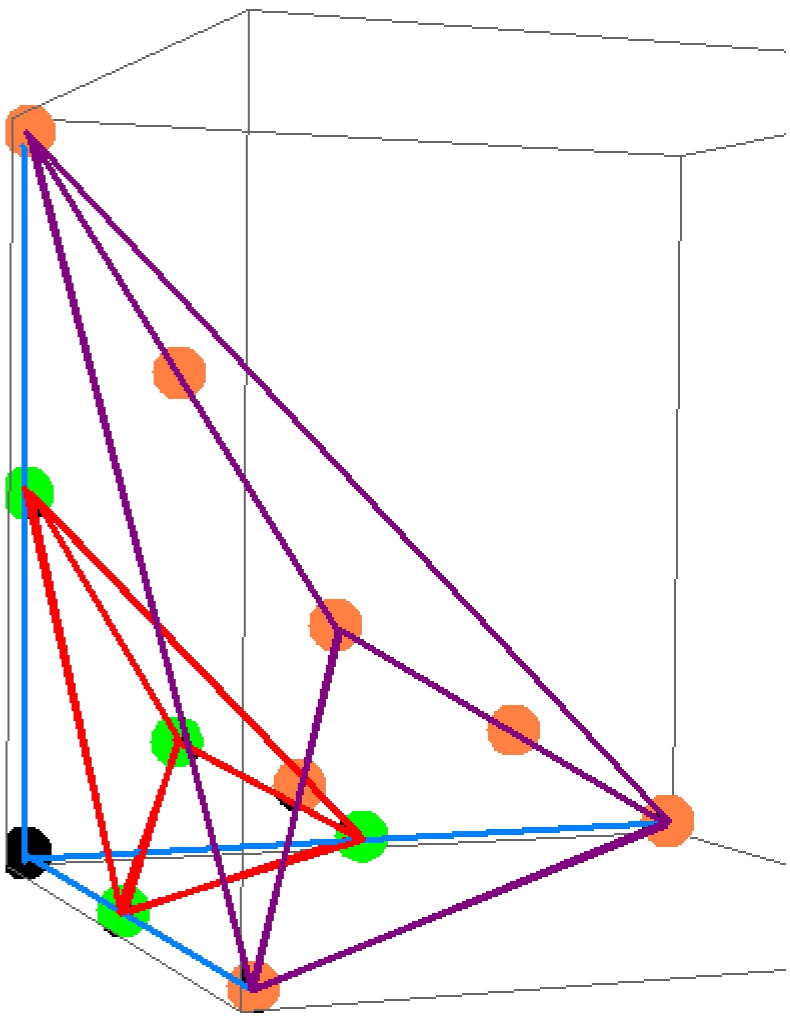
- If the sum of the entries of each generator of a semigroup adds to the same number *d*, then the semigroup is said to be *homogeneous* and of degree *d*.

- For example, the homogeneous semigroup <(2,0),(1,1) > is of degree 2, and the homogeneous semigroup <(5,0,0),(0,5,0),(0,0,5),(1,2,2)> is of degree 5.

- I am looking in particular at *simplicial* homogenous semigroups:
  - Those with a lattice in the plane and with (d,0) and (0,d) in its set of generators.
  - Those with a lattice in space and with (d,0,0), (0,d,0), and (0,0,d) in its set of generators.

The second example above is also an example of a simplicial homogeneous semigroup (which is what I will mean by "semigroup" from here on).

S=<(3,0,0),(0,3,0),(0,0,3),(1,1,1)>



- Unfortunately, a semigroup lattice in 3-space can be quite confusing.
- Here's part of one for

S=<(3,0,0),(0,3,0),(0,0,3),(1,1,1)>

(The colored lines and dots are an attempt to make things less confusing. Tiers are connected.)

Semigroup Lattice Revisited

- Once again, here is part of the semigroup lattice for $S=<(2,0),(1,1),(0,2)>$
- The following non-generator elements appear since

  (0,4) = (0,2) + (0,2)

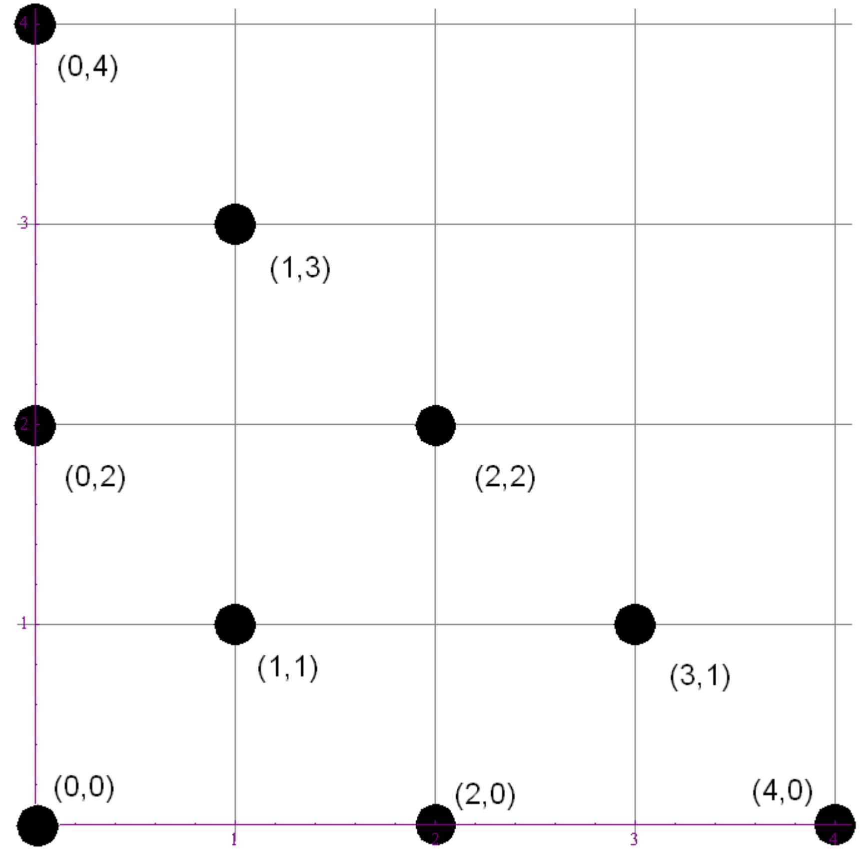
  (1,3) = (1,1) + (0,2)

  (2,2) = (1,1) + (1,1)

  (3,1) = (1,1) + (2,0)

  (4,0) = (2,0) + (2,0)

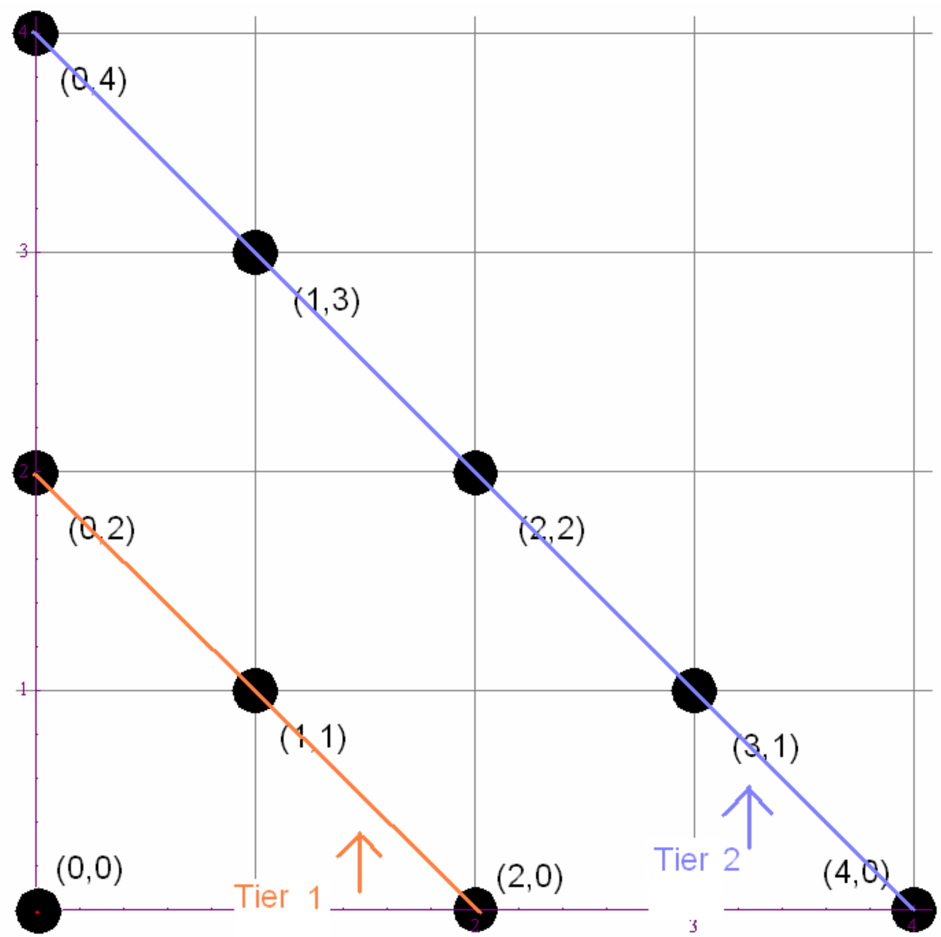
  are all in the semigroup. The element (0,0) is also in the semigroup.



Tier Structure

- Note that a “tier” structure emerges.

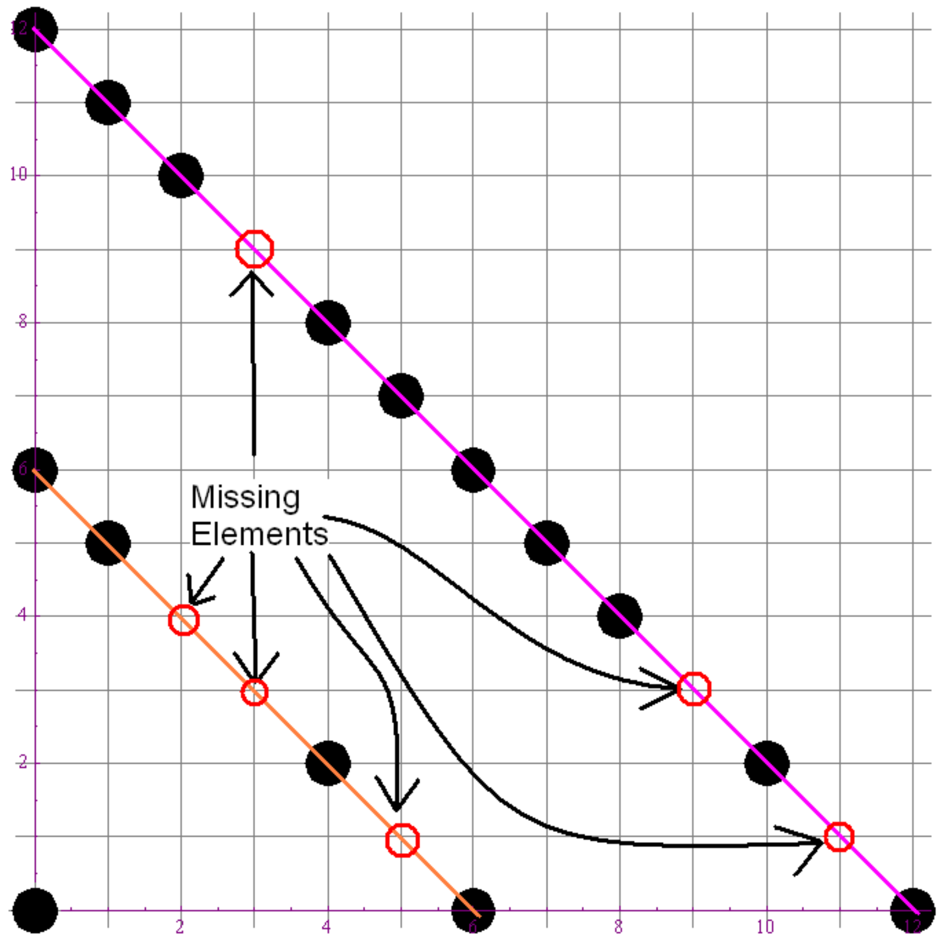
- For a semigroup of degree *d*, an element with entries that add to *nd* is said to be on the *n*th tier.



Missing Elements

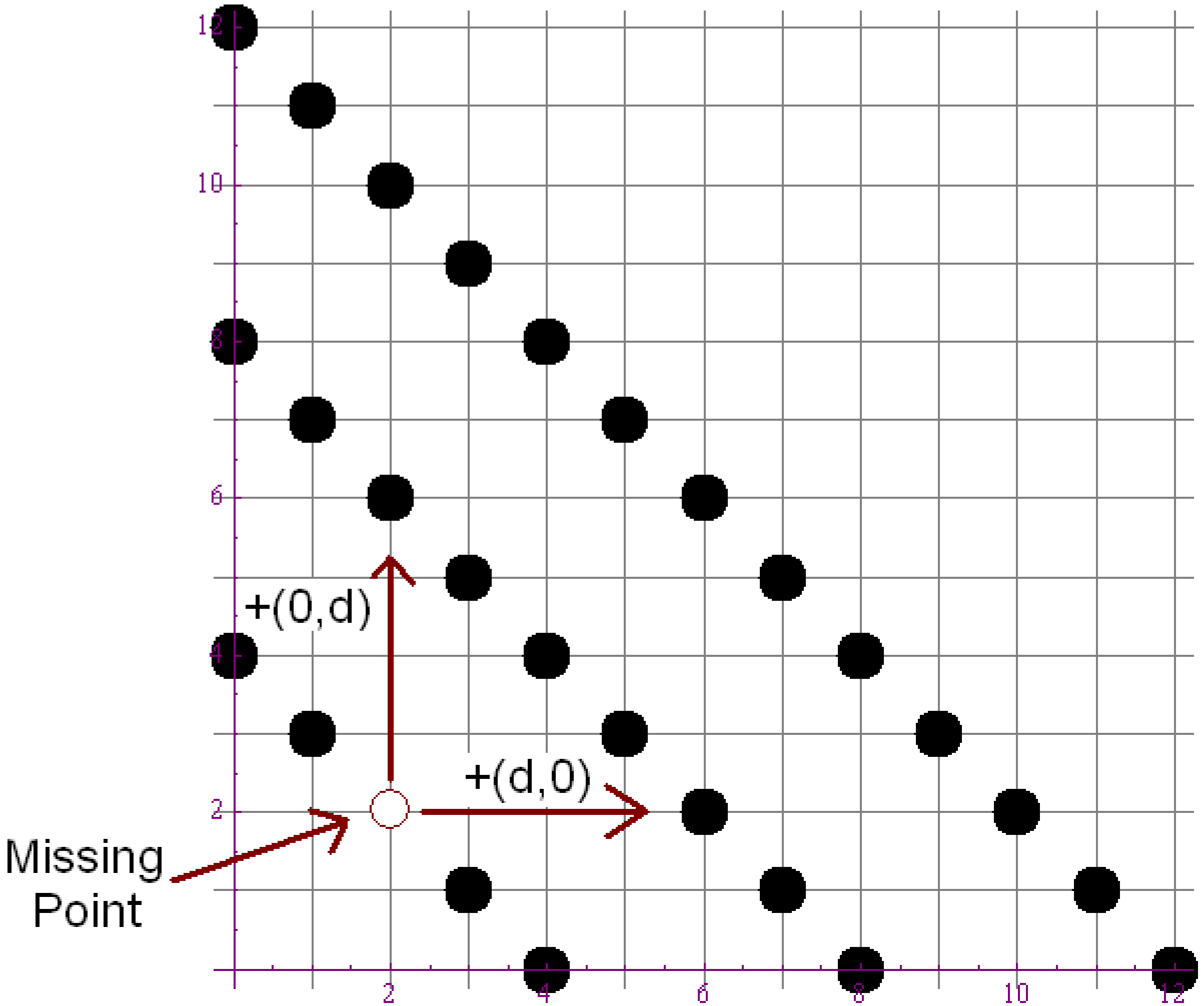
- A simplicial homogenous semigroup of degree *d* may be “missing” some elements or points—that is, there may be tuples with entries that add up to a multiple of *d* that do not belong to the semigroup

- The example on the right is for S=<(0,6),(1,5),(4,2),(6,0)>
  - The missing elements on the first tier are (2,4), (3,3), and (5,1)
  - The missing elements on the second tier are (3,9), (9,3), and (11,1)



Cohen-Macaulay Semigroups with Lattice in the Plane

- A simplicial homogeneous semigroup S of degree d, with lattice in the plane, is said to be Cohen-Macaulay (CM) if there do **not** exists any points p such that:
  - p is not in S—i.e. p is a missing point—and $p+(d,0)$ and $p+(0,d)$ are both in S
  - Otherwise S is said to be non-Cohen-Macaulay (NCM).

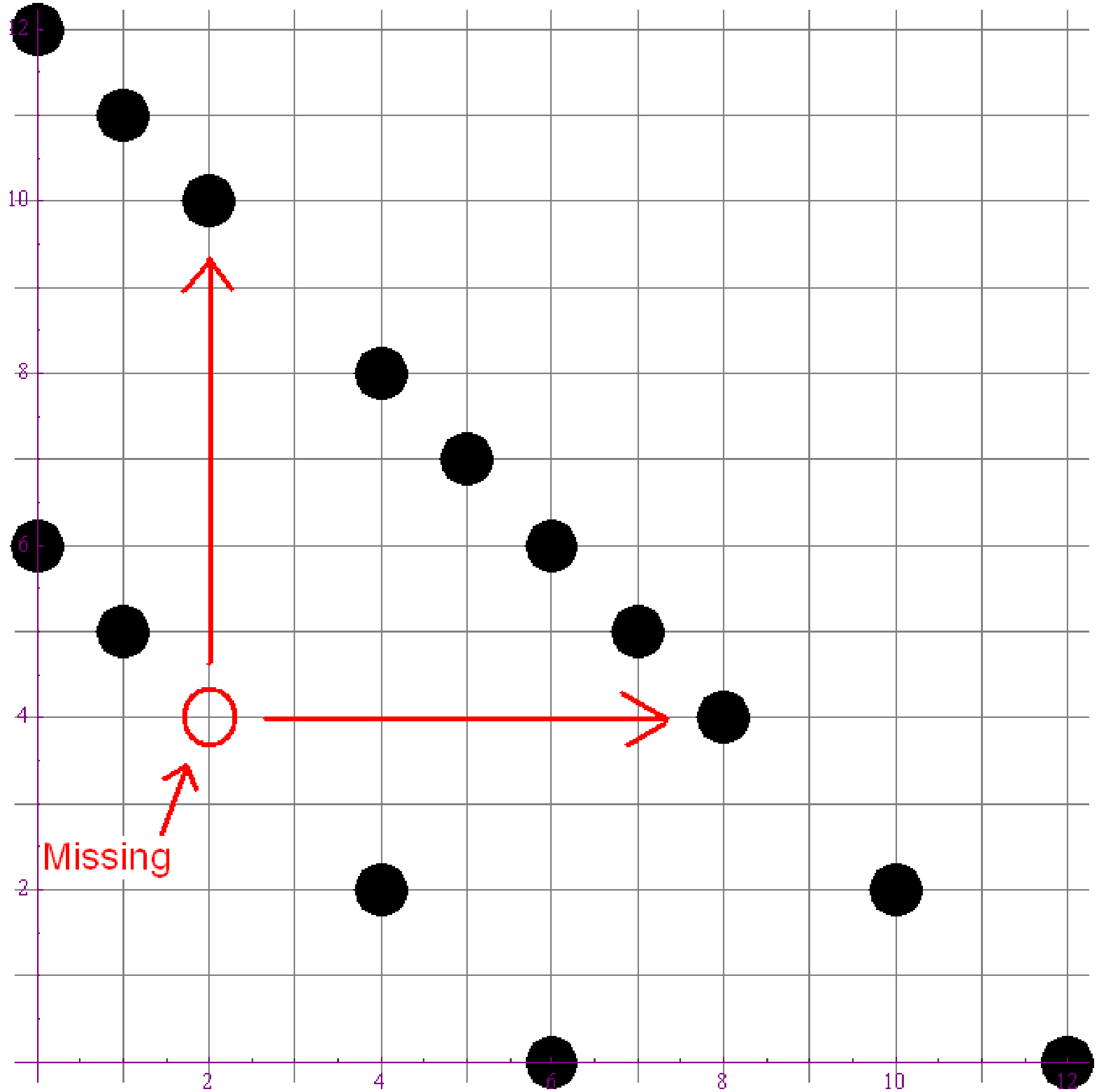


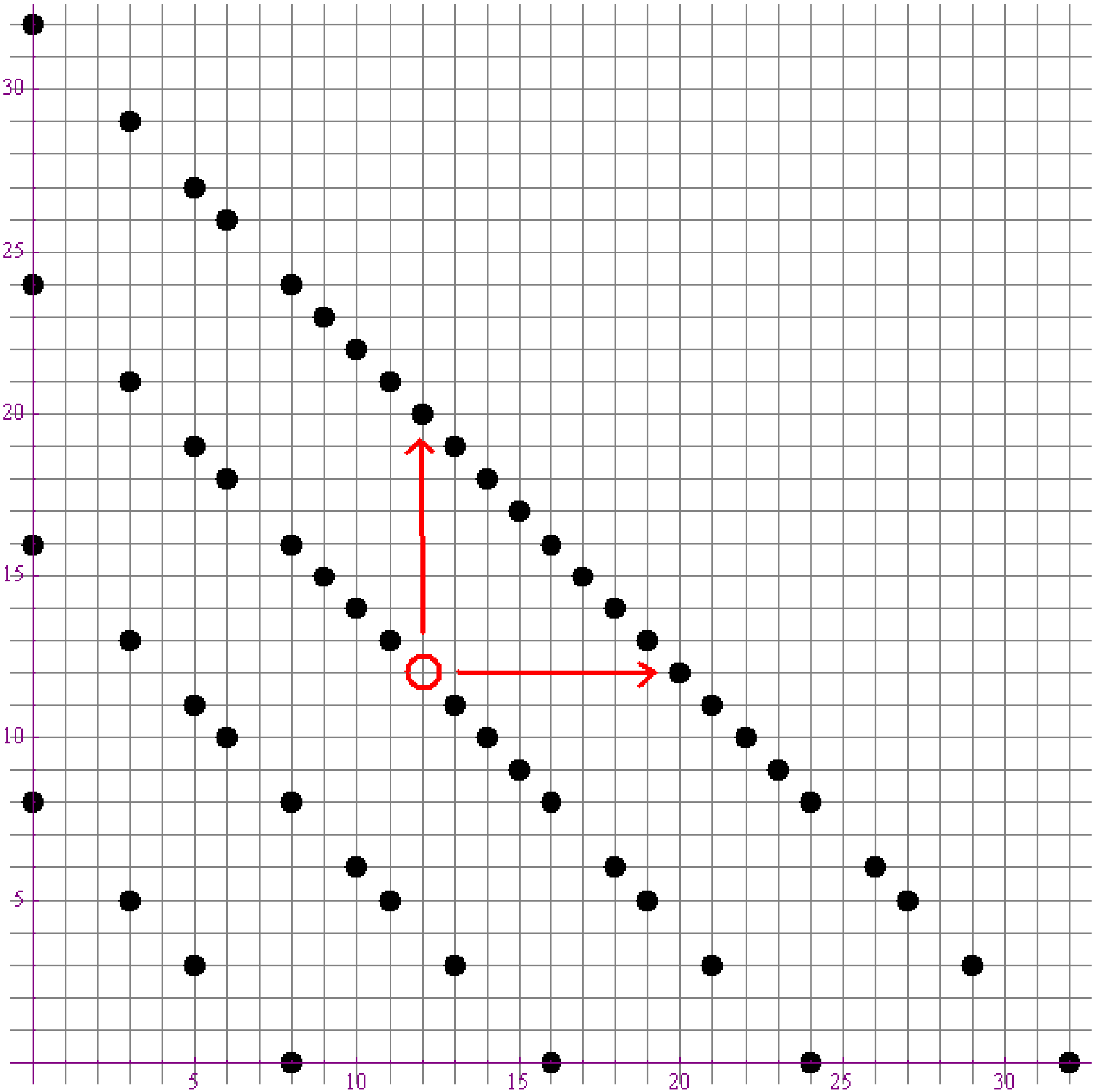
- S=<(0,4), (1,3), (3,1), (4,0)>
- (2,2) is missing, but (6,2)=(3,1)+(3,1) and (2,6)=(1,3)+(1,3) are not, so S is NCM.

S=<(0,6),(1,5),(4,2),(6,0)>   (2,4) is missing

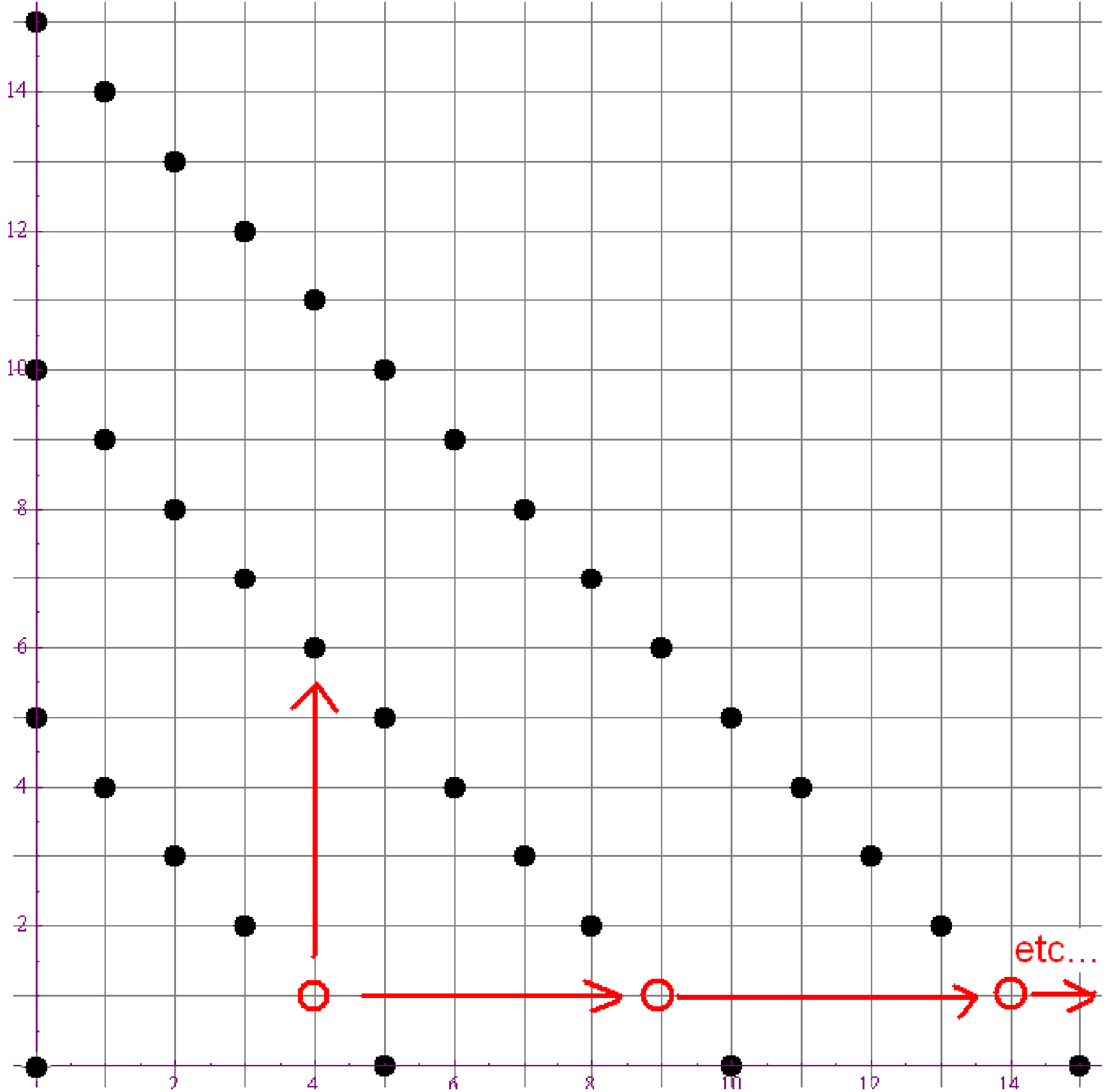
but (8,4) = (4,2) + (4,2) and (2,10) = (1,5) + (1,5) are not

So S is NCM

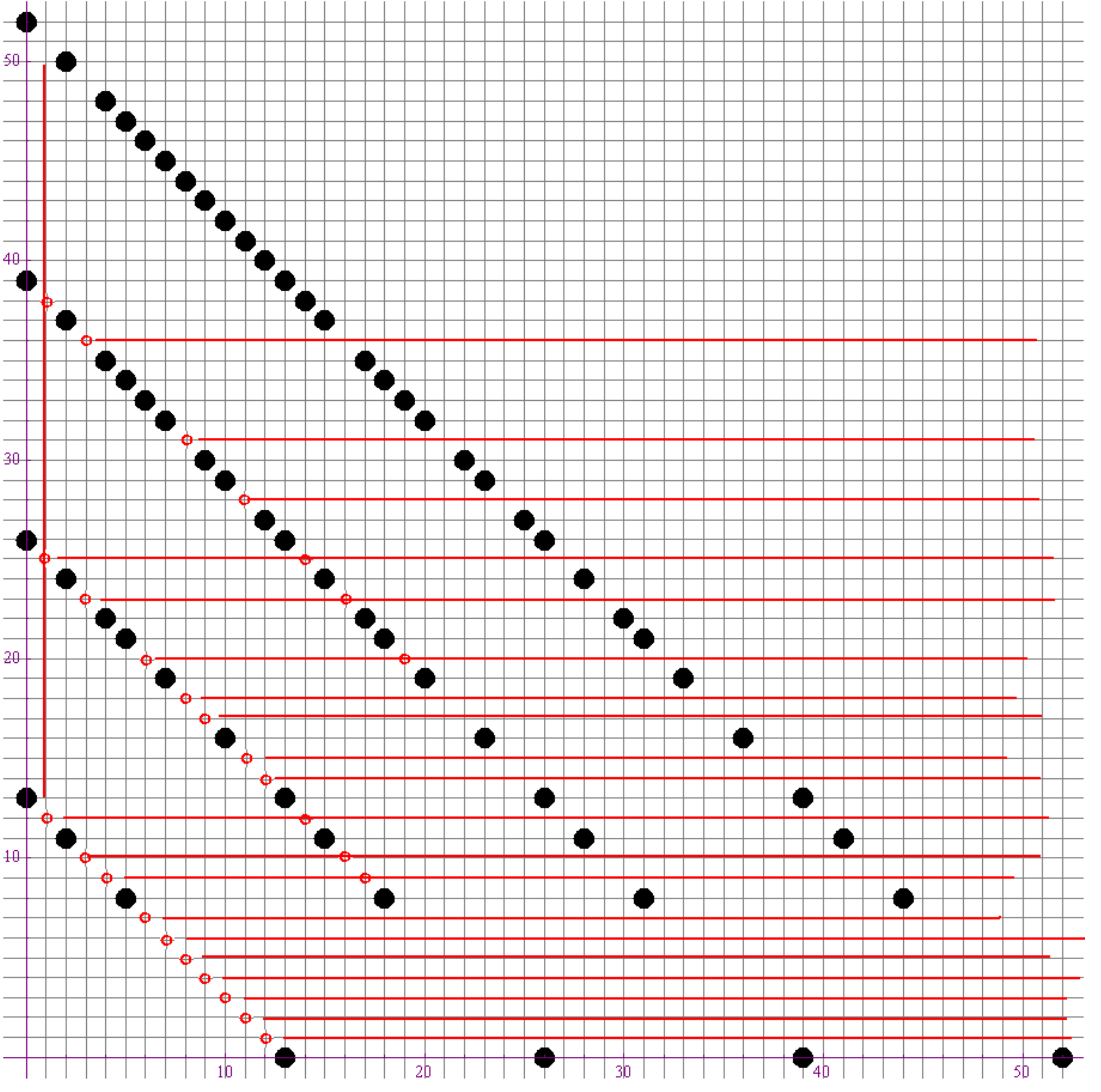




- *S* = <(0,8), (3,5), (5,3), (8,0)>
Missing point: (12,12)
(20,12) = 4(3,5) and (12,20) = 4(5,3) are not missing, so *S* is NCM



- *S* = <(0,5),(1,4),(2,3),(3,2),(5,0)> is CM since adding (d,0) to any of the missing points gives another missing point.



- *S*=<(0,13),(2,11),(5,8)(13,0)> is also CM but this is harder to see (and show): For each missing point p, either p+(d,0), or p+(0,d) is in *S*.

How Many Cohen-Macaulay Semigroups are There?

- Dr. Reid showed that for semigroups with a lattice in the plane, in the grand scheme of things Cohen-Macaulay semigroups are rare.
- $\#CM_2(d)$ is the number of Cohen-Macaulay semigroups of degree d with lattice in 2-space.
- $\#T_2(d)$ is the total number of (simplicial homogeneous) semigroups of degree d with lattice in 2-space.

$$\lim_{d \to \infty} \frac{\#CM_2(d)}{\#T_2(d)} = 0$$

- These numerical results from Dr. Reid's paper show that fairly quickly the numbers begin to favor NCM semigroups.
- ***d*** – degree
- ***#T(d)*** – total number number of semigroups
- ***#CM(d)*** – number of CM semigroups
- ***#NCM(d)*** – number of NCM semigroups

- For example, *#CM*(18)/*#T*(18)=0.0413647 so about 4.1 percent of semigroups of degree 18 are Cohen-Macaulay

d	$\#T(d)$	$\#CM(d)$	$\#NCM(d)$
1	1	1	0
2	1	1	0
3	3	3	0
4	6	5	1
5	15	12	3
6	27	16	11
7	63	37	26
8	120	51	69
9	252	97	155
10	495	142	353
11	1023	257	766
12	2010	359	1651
13	4095	647	3448
14	8127	920	7207
15	16365	1605	14760
16	32640	2266	30374
17	65535	3795	61740
18	130788	5410	125378

Semigroups With Lattice in 3-Space

- A simplicial homogeneous semigroup *S* of degree *d*, with lattice in 3-space, is said to be Cohen-Macaulay (CM) if there do **not** exists any points *p* such that:

  p is a missing point but none of *p*+(*d*,*d*,0), *p*+(*d*,0,*d*), and are *p*+(0,*d*,*d*) missing.

  Otherwise *S* is said to be non-Cohen-Macaulay (NCM).

Extending Results?

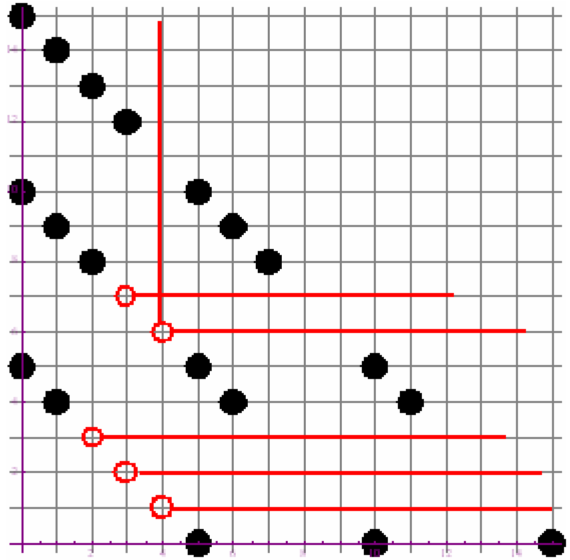
$$\lim_{d \to \infty} \frac{\#CM_3(d)}{\#T_3(d)} = 0$$

- Although numerically the evidence for the above statement is quite strong, it seems to be quite difficult to actually prove the statement using similar methods as the 2-space case due to several combinatorial complications.
- The numerical results shows that fewer than 4 percent of semigroups of degree 5 are Cohen-Macaulay. For degree 6+ the ratio starts to become a very small fraction.

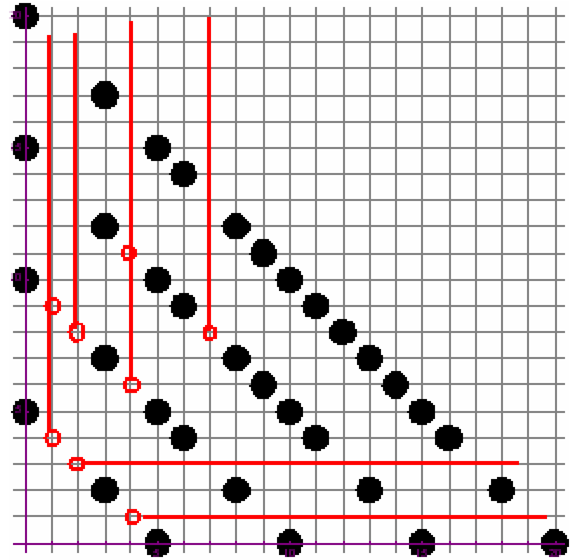
Back to the Plane

- Moving back to semigroups with a lattice in 2-space, what happens when we focus on semigroups with a specific number of generators?
- Note that since our semigroups are *simplicial* of degree d, we will always have at least the two generators $(d,0)$ and $(0,d)$.

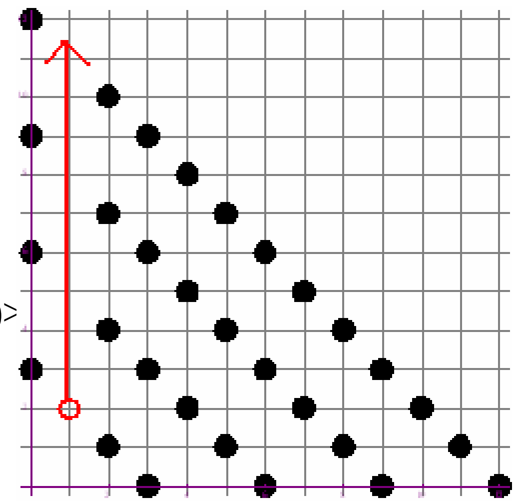
<(0,5),(1,4),(5,0)>



<(0,5),(3,1),(5,0)>



<(0,3),(2,1),(3,0)>



- A geometric argument (omitted) shows that for three generators— i.e. $S=<(0,d),(a,d-a),(d,0)>$ —S is always Cohen-Macaulay. Above are some examples.

$S = \langle(0,d),(b,d-b),(a,d-a),(d,0)\rangle$

- What about four generators? Note that since S is homogeneous we can just write the first entry for each generator since the second entry is then determined: that is, write $S=\langle 0,b,a,d\rangle$ instead of $S=\langle(0,d),(b,d-b),(a,d-a),(d,0)\rangle$

Holding *b* and *a* Constant

- Using brute force method we can figure out which semigroups are CM.
- Holding *d* constant and trying to see which combinations of *b* and *a* yield CM (or NCM) semigroups was not very enlightening.
- However, when I held both *b* and *a* constant and looked at which values of *d* caused the semigroup to be NCM, a pattern emerged.
- Below is a table of combinations of *b*, *a* and *d* that are NCM. For each entry, the top number is *b*, the middle is *a*, and the bottom is *d*.
- The largest value of *d* in each row is: 4, 9, 16, 25, 36, 49,… these are the perfect squares. In each case the largest value for *d* is $(a\text{-}1)^2$

b=1, a=3

1
3
4

b=1, a=4

1	1	1
4	4	4
5	6	9

b=1, a=5

1	1	1	1	1	1
5	5	5	5	5	5
6	7	8	11	12	16

b=1, a=6

1	1	1	1	1	1	1	1	1	1
6	6	6	6	6	6	6	6	6	6
7	8	9	10	13	14	15	19	20	25

b=1, a=7

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
8	9	10	11	12	15	16	17	18	22	23	24	29	30	36

b=1, a=8

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8	8
9	10	11	12	13	14	17	18	19	20	21	25	26	27	28	33	34	35	41	42	49

First Result

If S is a monomial curve generated by

$$\{(0,d),(b,d-b),(a,d-a),(d=(a-b)i-bj=ai-b(i+j),0)\}$$

$$=\underline{\{0,b,a,d\}}$$

$$\text{where}\begin{cases}\underline{gcd(b,a)}=1\\ b+1\le i\le a-1\\ 0\le j\le a-1-i\end{cases}$$

then S is not Cohen-Macaulay.

Similar kinds of observations paved the way for the discovery of the above result concerning when some of these four-generator NCM semigroups. Note the limitation that *gcd*(*b*,*a*)=1.

Other Results

- Extending the first result, here is a table of the number of NCM semigroups generated by 4 elements with *gcd*(*a*,*b*)=1 that I have accounted for compared to the actual number found using brute force calculations.

Degree	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Actual # of NCM	1	1	3	3	6	9	9	11	18	19	22	26	32	36	43	46	57	58	63	69	82	94	96	103	112	120	137
# Accounted For	1	1	3	3	6	9	9	11	18	19	21	26	32	36	41	44	56	57	61	67	78	91	91	97	108	116	130
# Unnaccounted For	0	0	0	0	0	0	0	0	0	0	1	0	0	0	2	2	1	1	2	2	4	3	5	6	4	4	7