A Generalization of Recursive Integer Sequences of Order 2

Stephen A. Parry
Missouri State REU
August 1, 2007

Synopsis

- A ring extension from Ω to Ω_α.
- The definition of Period-1 and Period-2 orbits, along with an algorithm for finding Period-2 matrices.
- Discuss isomorphisms
- Generalized Relations
- Powers of Period-1 and Period-2 2x2 Matrices
- Multiplying NxM Recursive Matrices
- Generalizations
- Problems that need additional research

Definitions

Definition: A recursive integer sequence of order 2 is a sequence in the form $A_n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbf{Z}$.

Definition: The Fibonacci sequence is a recursive sequence

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$$

$$\{\ldots 1, 1, 2, 3, 5, 8, \ldots\}$$

Definition: The Lucas sequence is a recursive sequence

$$L_n = L_{n-1} + L_{n-2}, \quad L_1 = 1, \quad L_2 = 3.$$

$$\{\ldots 1, 3, 4, 7, 11, 18, \ldots\}$$

Definition: $\Omega = \left\{ \begin{bmatrix} a+b & b \\ b & a \end{bmatrix} \in \mathrm{M}(2, \mathbf{Z}) \right\}$

Recursive Matrix

- We will use this definition of a recursive matrix interchangeably with the recursive sequence representation.

Definition: A recursive matrix is an nxm matrix in the form

$$\begin{bmatrix} A_n & A_{n-1} & \mathrm{L} & A_{n-c} \\ A_{n-1} & \mathrm{N} & \mathrm{N} & \mathrm{N} \\ \mathrm{M} & \mathrm{N} & \mathrm{N} & \mathrm{M} \\ A_{n-r} & \mathrm{L} & \mathrm{K} & A_{n-r-c} \end{bmatrix}$$

Note: The recursive matrix need not be square.

Recursive Matrices

Example: $A_n = 2A_{n-1} + 4A_{n-2} \quad A_1 = 1,\ A_2 = 2$

$\{\ldots 1, \underline{2, 8, 24}, \ldots\}$ $\quad$ $\{\ldots \underline{1, 2, 8, 24}, \ldots\}$

$$\begin{bmatrix} 24 & 8 \\ 8 & 2 \end{bmatrix} \qquad \begin{bmatrix} 24 & 8 & 2 \\ 8 & 2 & 1 \end{bmatrix}$$

Example: $A_n = A_{n-1} + A_{n-2} \quad A_1 = 1,\ A_2 = 1$

$\{\ldots \underline{1, 1, 2}, 3, 5, 8, \ldots\}$ $\quad$ $\{\ldots \underline{1, 1, 2, 3, 5, 8, 13}, \ldots\}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 13 & 8 & 5 & 3 \\ 8 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

More Definitions

Definition: **F** is the set of recursive matrices with integer multiples of Fibonacci entries.

Definition: **L** is the set of recursive matrices with integer multiples of Fibonacci entries.

Examples:

$$3\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in \mathbf{F}$$

$$\begin{bmatrix} 18 & 11 & 7 & 4 \\ 7 & 4 & 3 & 1 \\ 4 & 3 & 1 & 2 \end{bmatrix} \in \mathbf{L}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbf{F}, \Omega$$

About Ω

Theorem: Ω forms an integral domain.

Definition: We define σ as the shift map.

Theorem:

$$\sigma^n = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \in \mathbf{F} \quad \forall\, n \in \mathbf{Z}$$

a

Theorem:

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \in \mathbf{L}$$

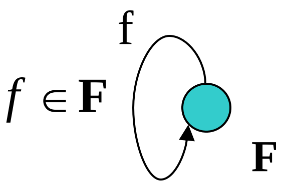
$$A^{2k} = 5^k \begin{bmatrix} F_{4k+1} & F_{4k} \\ F_{4k} & F_{4k-1} \end{bmatrix} \in \mathbf{F}$$

$$A^{2k+1} = 5^k \begin{bmatrix} L_{4k+3} & L_{4k+2} \\ L_{4k+2} & L_{4k+1} \end{bmatrix} \in \mathbf{L}$$

Examples

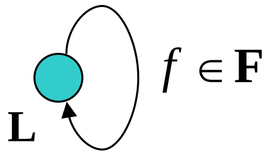
Fibonacci * Fibonacci

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix} \in \mathbf{F}$$



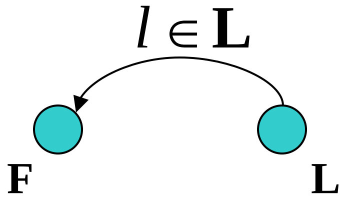
Fibonacci * Lucas

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 7 \\ 7 & 4 \end{bmatrix} \in \mathbf{L}$$



Lucas * Lucas

$$\begin{bmatrix} 7 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} = 5\begin{bmatrix} 8 & 5 \\ 5 & 3 \end{bmatrix} \in \mathbf{F}$$



Ring Extension

- Now we have a result for the Fibonacci and Lucas numbers, we would like to make a generalization for all recursive sequences,

$$A_n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbf{Z}.$$

- The matrix we would require in the 2x2 case is

$$\begin{bmatrix} \alpha b + \beta a & b \\ b & a \end{bmatrix}.$$

- Since the identity is in Ω, then we must have the identity in the ring extension. Thus, β=1.

- Therefore, we will concentrate on

$$A_n = \alpha A_{n-1} + A_{n-2}, \quad \alpha \in \mathbf{Z}.$$

Ring Extension

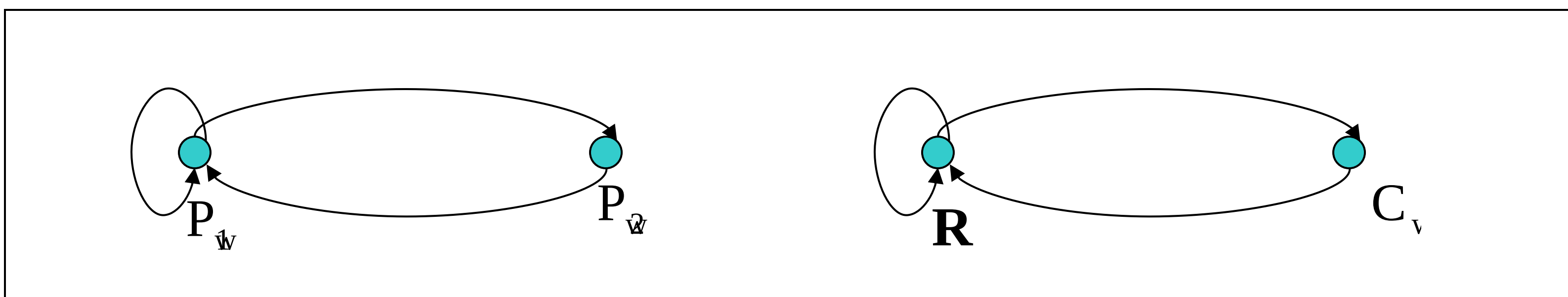
$$\Omega_\alpha = \begin{bmatrix} \alpha b + a & b \\ b & a \end{bmatrix}$$

$$\Omega = \begin{bmatrix} b + a & b \\ b & a \end{bmatrix}$$

More Definitions

Definition: A Period-1 sequence is any sequence when expressed in matrix form will be closed under multiplication. Define this set as P_{1w}.

Definition: A Period-2 sequence is any sequence when expressed in matrix form will be closed under multiplication in the union of the Period-2 sequence and its complimentary Period-1 sequence. Define this set as P_{2w}.



Isomorphism

Theorem: $\Omega_\alpha = \mathbf{Z}[\sigma_\alpha] \cong \mathbf{Z}[\phi], \quad \sigma_\alpha = \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}$

Theorem:

$$(1+\alpha\phi)^{2n}/\phi^2 - \alpha\phi - 1 = A_{2n-1} + A_{2n}\phi$$

such that

$$A_n = \alpha A_{n-1} + A_{n-2}, \quad A_1 = 1, A_2 = \alpha.$$

Example:

$$(1+\phi)^3/\phi^2 - \phi - 1 =$$

$$(1+3\phi+3\phi^2+\phi^3)/\phi^2 - \phi - 1$$

$$= 5 + 8\phi = F_5 + F_6\phi$$

Period-2 2x2 Matrices

Theorem: We express a Period-1 sequence as

$$\{\ldots 1, \alpha, \alpha^2+1, \alpha^3+2\alpha \ldots\}.$$

$$\Lambda_\alpha = \begin{bmatrix} \alpha^2+1 & \alpha \\ \alpha & 1 \end{bmatrix}.$$

Theorem: We express a Period-2 sequence for odd α as

$$\{\ldots \alpha, \alpha^2+2, \alpha^3+3\alpha \ldots\}.$$

$$\eta_\alpha = \begin{bmatrix} \alpha^3+3\alpha & \alpha^2+2 \\ \alpha^2+2 & \alpha \end{bmatrix}.$$

Theorem: We express a Period-2 sequence for even α as

$$\frac{1}{2}\{\ldots \alpha, \alpha^2+2, \alpha^3+3\alpha \ldots\}.$$

$$\eta_\alpha = \frac{1}{2}\begin{bmatrix} \alpha^3+3\alpha & \alpha^2+2 \\ \alpha^2+2 & \alpha \end{bmatrix}$$

Proof for odd α

Definition: Given a recursive sequence in the form, $A_n = \alpha A_{n-1} + A_{n-2}$

we define the characteristic polynomial, $x^2 = \alpha x + 1.$

$$x = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}.$$

$$\phi = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2}.$$

Proof:

$$\phi^2 = \alpha\phi + 1$$

$$\sqrt{\alpha^2 + 4}\phi^2 = \sqrt{\alpha^2 + 4}(\alpha\phi + 1)$$

$$= (\alpha^2 + 2)\phi + \alpha$$

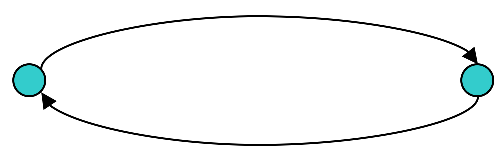
Note: The discriminant of the characteristic polynomial plays an important role.

Proof for odd α

$$\eta_\alpha = \begin{bmatrix} \alpha^3 + 3\alpha & \alpha^2 + 2 \\ \alpha^2 + 2 & \alpha \end{bmatrix}$$

$$\eta_\alpha^2 = \begin{bmatrix} \alpha^6 + 7\alpha^4 + 13\alpha^2 + 4 & \alpha^5 + 6\alpha^3 + 8\alpha \\ \alpha^5 + 6\alpha^3 + 8\alpha & \alpha^4 + 5\alpha^2 + 4 \end{bmatrix}$$

$$= (\alpha^2 + 4) \begin{bmatrix} \alpha^4 + 3\alpha^2 + 1 & \alpha^3 + 2\alpha \\ \alpha^3 + 2\alpha & \alpha^2 + 1 \end{bmatrix} = (\alpha^2 + 4)\Lambda_\alpha^2$$



α=5

Example:

Period-1

$$\{\ldots 1, 5, 26, \ldots\}.$$

$$\Lambda_5 = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 701 & 135 \\ 135 & 26 \end{bmatrix}$$

Period-2

$$\{\ldots 5, 27, 140 \ldots\}.$$

$$\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}.$$

$$\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$$

$$= 29\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$$

$$= 29\begin{bmatrix} 3775 & 727 \\ 727 & 140 \end{bmatrix}$$

Generalized Recursive Relations

Definition: $\sigma_\alpha = \begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbf{Z}$.

Theorem: $\sigma_1^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

Theorem: $\sigma_\alpha^n = \begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{bmatrix}$

Theorem: $A_{n+m} = A_{n-1}A_m + A_n A_{m+1}$

Proof: $\sigma_\alpha^m \sigma_\alpha^{n-1} = \sigma_\alpha^{m+n-1}$.

Theorem: $A_{2k-1}^2 + \alpha A_{2k} A_{2k-1} - A_{2k}^2 = 1$

Proof: $\det(\Lambda_\alpha^n \Lambda_\alpha) = 1$.

The Shift Map

Example: $\alpha = 2.$

$$\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

Period-1

$$\{\ldots 0,1,\underline{2,5,}12\ldots\}$$

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 5 & 2 \end{bmatrix}$$

$$\{\ldots 0,1,\underline{2,5,12}\ldots\}$$

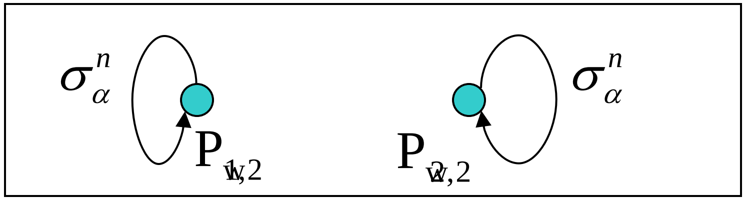
Period-2

$$\{\ldots 0,\underline{1,3,}7,17\ldots\}$$

$$\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 7 \\ 7 & 3 \end{bmatrix}$$

$$\{\ldots 0,1,\underline{3,7,17}\ldots\}$$



A Characteristic of Period-1 Matrices

- Period-1 Matrices act as units to their correlating Period-2 Matrices

Theorem: $Let\ \ \delta \in E \subseteq \Omega_{\alpha},\ then\ \ \delta\sigma_{\alpha}^{n} \in E\ \ \forall\ \alpha \in \mathbf{Z}$

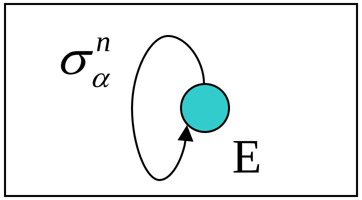
Proof:

$$\delta = \begin{bmatrix} \alpha b + a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$$

$$\delta\sigma_{\alpha} = \begin{bmatrix} \alpha^2 b + \alpha a + b & \alpha b + a \\ \alpha b + a & b \end{bmatrix} = \begin{bmatrix} B_4 & B_3 \\ B_3 & B_2 \end{bmatrix}$$

M

$$\delta\sigma_{\alpha}^{n} = \begin{bmatrix} B_{n+3} & B_{n+2} \\ B_{n+2} & B_{n+1} \end{bmatrix}.$$



Powers of Λ_α and η_α

Theorem: $$\Lambda_\alpha^n = \begin{bmatrix} A_{2n+1} & A_{2n} \\ A_{2n} & A_{2n-1} \end{bmatrix}$$

Theorem: For α even

$$\eta_\alpha^{2n-1} = \left(1+\frac{\alpha^2}{4}\right)^{n-1} \eta_\alpha \Lambda_\alpha^{2n-2}$$

$$\eta_\alpha^{2n} = \left(1+\frac{\alpha^2}{4}\right)^{n} \Lambda_\alpha^{2n}$$

Theorem: For α odd

$$\eta_\alpha^{2n-1} = \left(\alpha^2+4\right)^{n-1} \eta_\alpha \Lambda_\alpha^{2n-2}$$

$$\eta_\alpha^{2n} = \left(\alpha^2+4\right)^{n} \Lambda_\alpha^{2n}$$

Periodicity

Theorem: $If\ C^2 \in \Lambda_\alpha^n, then\ C \in \Lambda_\alpha^n, or\ C \in \eta_\alpha^n$

Proof:

$$C = \begin{bmatrix} \alpha b + a & b \\ b & a \end{bmatrix}$$

$$C^2 = \begin{bmatrix} (\alpha b + a)^2 + b^2 & \alpha b + 2ab \\ \alpha b + 2ab & b^2 + a^2 \end{bmatrix}$$

$$X = (\alpha b + a)^2 + b^2$$

$$Y = \alpha b + 2ab$$

$$Z = b^2 + a^2$$

$$X > 0, Z > 0,$$

$$let \quad \alpha b + 2ab = 0$$

$$b(\alpha b + 2a) = 0$$

$$b = 0, b = \frac{-2a}{\alpha}$$

$$If\ b = 0,$$

$$C = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \Lambda_\alpha^n$$

$$If\ b = \frac{-2a}{\alpha},$$

$$C = -a\begin{bmatrix} \alpha^2 + 3 & \frac{\alpha^2 + 2}{\alpha} \\ \frac{\alpha^2 + 2}{\alpha} & 1 \end{bmatrix}$$

$$a = k\alpha \longrightarrow \eta_\alpha, \alpha\ odd$$

$$a = k\frac{\alpha}{2} \longrightarrow \eta_\alpha, \alpha\ even$$

4x4 Period-1 and Period-2 Matrices

Example:

$$\begin{bmatrix} 13 & 8 & 5 & 3 \\ 8 & 5 & 3 & 2 \\ 5 & 3 & 2 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} F_7 & F_6 & F_5 & F_4 \\ F_6 & F_5 & F_4 & F_3 \\ F_5 & F_4 & F_3 & F_2 \\ F_4 & F_3 & F_2 & F_1 \end{bmatrix} = \left[\begin{array}{c|c} \Lambda_\alpha^3 & \Lambda_\alpha^2 \\ \hline \Lambda_\alpha^2 & \Lambda_\alpha^1 \end{array}\right]$$

$$B_1 = \begin{bmatrix} 29 & 18 & 11 & 7 \\ 18 & 11 & 7 & 4 \\ 11 & 7 & 4 & 3 \\ 7 & 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} L_7 & L_6 & L_5 & L_4 \\ L_6 & L_5 & L_4 & L_3 \\ L_5 & L_4 & L_3 & L_2 \\ L_4 & L_3 & L_2 & L_1 \end{bmatrix} = \left[\begin{array}{c|c} \eta_\alpha \Lambda_\alpha^2 & \eta_\alpha \Lambda_\alpha^1 \\ \hline \eta_\alpha \Lambda_\alpha^1 & \eta_\alpha \Lambda_\alpha^0 \end{array}\right]$$

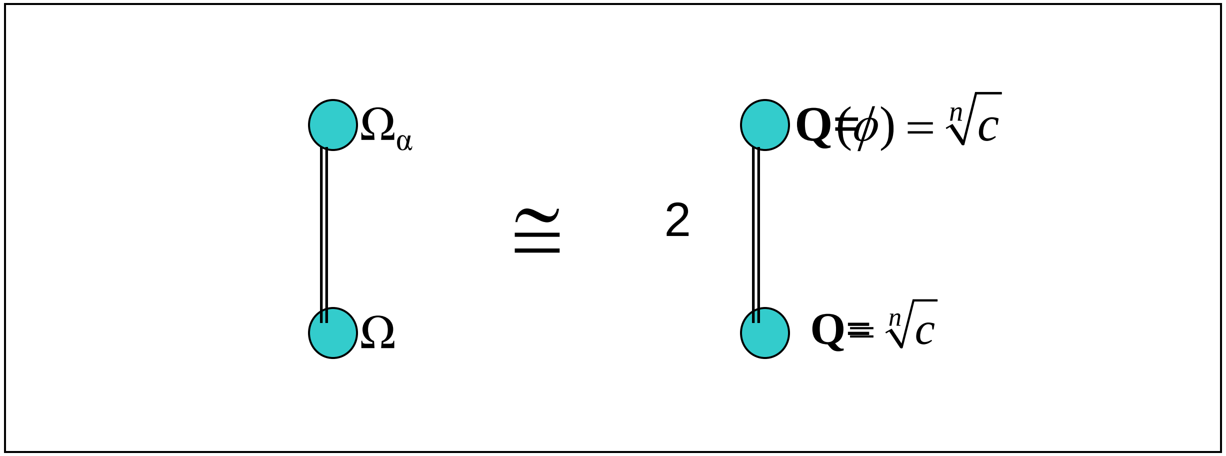
$$B_1^2 = (5)(3) \begin{bmatrix} 89 & 55 & 34 & 21 \\ 55 & 34 & 21 & 13 \\ 34 & 21 & 13 & 8 \\ 21 & 13 & 8 & 5 \end{bmatrix} = (5)(3) \left[\begin{array}{c|c} \Lambda_\alpha^5 & \Lambda_\alpha^4 \\ \hline \Lambda_\alpha^4 & \Lambda_\alpha^3 \end{array}\right]$$

Note: Similar to Period-1 and Period-2 2x2 matrices, it is possible to create a general formula for every α.

Higher Degree Periods

Theorem: $For\ n = 1, 2\ \exists\ a, b, c \in \mathbf{Z}$ such that the primitive case of $(a + b\phi)^n = c\phi^n$ is true.

Proof: The proof is dependent on the fact the degree of the ring extension is 2.



Thus, there fails to exist periods of degree greater than 2.

The NxM case

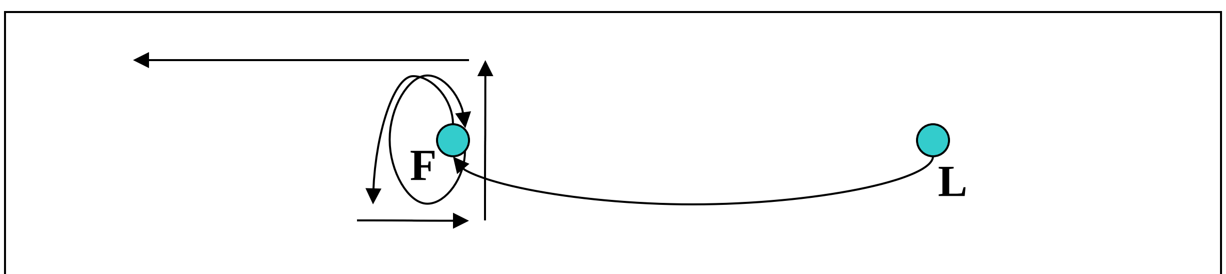
Definition: A complete orbit is an orbit closed under $\mathrm{P}_1 \cup \mathrm{P}_2$.

Example:

$$C_1 = \begin{bmatrix} 11 & 7 \\ 7 & 4 \\ 4 & 3 \end{bmatrix} \in \mathbf{L}$$

$$C_1 C_1^T = 5 \begin{bmatrix} 34 & 21 & 13 \\ 21 & 13 & 8 \\ 13 & 8 & 5 \end{bmatrix} \in \mathbf{F}$$

$$C_1 C_1^T C_1 = 5 \begin{bmatrix} 573 & 361 \\ 354 & 223 \\ 219 & 138 \end{bmatrix} \notin \mathbf{F}, \mathbf{L}$$



This complete one orbit, not two; this fails to be a complete orbit.

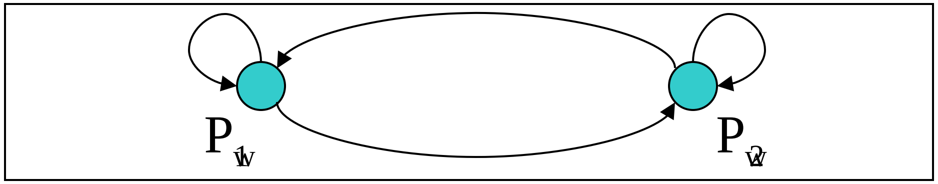
Resulting Generalizations

Theorem: We are guaranteed a complete orbit when we are given nxm matrix where $n \in \mathbf{Z}, m \in 2\mathbf{Z}.$

Theorem: Every nxn recursive matrix, M_n, where n is even , forms a ring.

Theorem: The set of nxm matrices that form a complete orbit is a semigroup.

Theorem: $If\ C^2 \in \mathbf{P}_1, then\ C \in \mathbf{P}_1, or\ C \in \mathbf{P}_2$



Problems of Interest

- Relations between Period-1 and Period-2 Sequences:

$$(B_n + B_{n+1}\phi)^{2k} = \gamma^k (A_1 + A_2\phi)^{k(n+1)}.$$

- Finding more isomorphisms
  - Continued fraction maps
  - Eigenvalue maps
  - Determinant maps
- Forming relationships for any power of every nxm recursive matrix in the Period-1 and Period-2 sets.
- Studying recursive relations of greater order.

References

- [1] George E. Andrews. Number Theory. Dover, 1994.
- [2] Lin Dazheng. Fibonacci matrices. Fibonacci Quarterly, 37(1):14-20, 1999.
- [3] Michele Elia. A note on fibonacci matrices of even degree. International Journal of Mathematics and Mathematical Sciences, 27(8):457-469, 2001.
- [4] John B. Fraleigh. A First Course in Abstract Algebra. Addison-Wesley, seventh edition edition, 2003.
- [5] Ross Honsberger. Mathematical Gems III. MAA, 1985.
- [6] T.F. Mulcrone. Semigroup examples in introductory modern algebra. The American Mathematical Monthly, 69(4):296{301, Apri., 1962.
- [7] N.N. Vorobyov. The Fibonacci Numbers. The University of Chicago, 1966.
- [8] Lawrence C. Washington. Some remarks on fibonacci matrices. Fibonacci Quarterly, 37(4):333-341, 1999.
- [9] Kung-Wei Yang. Fibonacci with a golden ring. Mathematics Magazine, 70(2):131-135, Apr.,1997.