A Generalization of Recursive Integer Sequences of Order 2

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Synopsis

- \circ A ring extension from Ω to Ω_{α}
- The definition of Period-1 and Period-2 orbits, along with an algorithm for finding Period-2 matrices.
- Discuss isomorphisms
- Generalized Relations
- Powers of Period-1 and Period-2 2x2 **Matrices**
- Multiplying NxM Recursive Matrices
- Generalizations
- Problems that need additional research

Definitions

Definition: A recursive integer sequence of order 2 is a sequence in the form $A_n = \alpha A_{n-1} + \beta A_{n-2}$, $\alpha, \beta \in \mathbb{Z}$.

1S

ursive integer sequence of order 2 is a se
 $A_n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}$.

ibonacci sequence is a recursive sequence
 $F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$
 $\{\dots, 1, 1, 2, 3, 5, 8, \dots\}$

ucas sequence is a r **S**

sive integer sequence of or
 $n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in$

somacci sequence is a recur
 $F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1,$
 $\{\ldots 1, 1, 2, 3, 5, 8, \ldots\}$ S

ve integer sequence of order 2 is a sequence in the
 $= \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}$.

nacci sequence is a recursive sequence
 $= F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$

{...1,1,2,3,5,8,...}

as sequence is a recursive s **Figure 12**
 F_{*n*} = $\alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}$.
 F *n* = $F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1.$

{...1,1,2,3,5,8,...}
 F *n* = $L_n = L_{n-1} + L_{n-2}, \quad L_1 = 1, \quad L_2 = 3.$

{...1,3,4,7,11,18,...}
 $\begin{bmatrix} a+b & b \end{bmatrix} \$ Definition: The Fibonacci sequence is a recursive sequence **Latter Latter Lat** ence of order 2 is a
 $\alpha, \beta \in \mathbb{Z}$.

is a recursive sequence
 $F_1 = 1, F_2 = 1$.
 $\{3, 8, ...\}$

recursive sequence
 $L_1 = 1, L_2 = 3$.
 $\{18, ...\}$
 $(2, \mathbb{Z})$
 $(2, \mathbb{Z})$ **S**

ive integer sequence of orde
 $= \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}$.

onacci sequence is a recursiv
 $\sum_{n=1}^{n} F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1, \quad F_3 = 1, \quad F_4 = 1, \quad F_5 = 1, \quad F_6 = 1, \quad F_7 = 1, \quad F_7 = 1, \quad F_8 = 1, \quad F_9 = 1, \quad F$ **b**
 c integer sequence of orde
 $\alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}$.

acci sequence is a recursiv
 $= F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F$

{...1,1,2,3,5,8,...}

i sequence is a recursive se
 $= L_{n-1} + L_{n-2}, \quad L_1 = 1, \quad L$

{...1,3, **tions**

A recursive integer sequence of order 2 is a sequence in the

form $A_n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}$.

The Fibonacci sequence is a recursive sequence
 $F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1$.
 $\{\ldots, 1, 1, 2, 3,$

Definition: The Lucas sequence is a recursive sequence

$$
L_n = L_{n-1} + L_{n-2}, \quad L_1 = 1, \quad L_2 = 3.
$$

{...1,3,4,7,11,18,...}

Definition:
$$
\Omega = \left\{ \begin{bmatrix} a+b & b \\ b & a \end{bmatrix} \in M(2, \mathbb{Z}) \right\}
$$

Recursive Matrix

 \circ We will use this definition of a recursive matrix interchangeably with the recursive sequence representation. **atrix**
 $\begin{bmatrix}\n\text{effinition of a recursive matrix} \\
\text{with the recursive sequence}\n\end{bmatrix}$
 $\begin{bmatrix}\n\text{atrix is an nxm matrix in the form} \\
\text{atrix is an nxm matrix in the form} \\
\begin{bmatrix}\n\text{atatrix} & A_{n-c} \\
\text{atatrix} & \text{atatrix}\n\end{bmatrix} \\
\text{atatrix}\n\begin{bmatrix}\n\text{atatrix} & \text{atatrix} \\
\text{atatrix} & \text{atatrix}\n\end{bmatrix} \\
\text{atatrix}\n\begin{bmatrix}\n\text{atatrix} & \text{atatrix} \\
\text{atatrix} & \text{atatrix}\n\end{b$

Definition: A recursive matrix is an nxm matrix in the form

give Matrix

\nuse this definition of a recursive matrix

\nnegably with the recursive sequence

\nstation.

\nA recursive matrix is an *nxm* matrix in the form

\n
$$
\begin{bmatrix}\nA_n & A_{n-1} & L & A_{n-c} \\
A_{n-1} & N & N & N \\
M & N & M & \\
A_{n-r} & L & K & A_{n-r-c}\n\end{bmatrix}
$$
\nreclusive matrix need not be square.

Note: The recursive matrix need not be square.

Recursive Matrices

More Definitions

Definition: F is the set of recursive matrices with integer multiples of

Fibonacci entries.

Definition: **L** is the set of recursive matrices with integer multiples of Definition: F is the set of recursive matrices with integer multiples of Fibonacci entries.

Fibonacci entries.

Examples:

More Definitions

\n**Definition: F** is the set of recursive matrices with integer multiples of Fibonacci entries.

\n**definition: L** is the set of recursive matrices with integer multiples of Fibonacci entries.

\n**Example:**

\n
$$
3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in \mathbf{F}
$$

\n
$$
\begin{bmatrix} 18 & 11 & 7 & 4 \\ 7 & 4 & 3 & 1 \\ 4 & 3 & 1 & 2 \end{bmatrix} \in \mathbf{L}
$$

\n
$$
\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbf{F}, \Omega
$$

About Ω

Theorem: Ω forms an integral domain.

Definition: We define σ as the shift map.

Theorem:

$$
\Omega \text{ forms an integral domain.}
$$
\n
$$
\mathbf{W} \text{ define } \sigma \text{ as the shift map.}
$$
\n
$$
\sigma^n = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \in \mathbf{F} \quad \forall \ n \in \mathbf{Z}
$$
\n
$$
\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \in \mathbf{L}
$$
\n
$$
A^{2k} = 5^k \begin{bmatrix} F_{4k+1} & F_{4k} \\ F_{4k} & F_{4k-1} \end{bmatrix} \in \mathbf{F}
$$
\n
$$
A^{2k+1} = 5^k \begin{bmatrix} L_{4k+3} & L_{4k+2} \\ L_{4k+2} & L_{4k+1} \end{bmatrix} \in \mathbf{L}
$$

Theorem:

$$
\Omega \text{ forms an integral domain.}
$$
\n
$$
\mathbf{W} \text{ define } \sigma \text{ as the shift map.}
$$
\n
$$
\sigma^n = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix}^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \in \mathbf{F} \quad \forall \ n \in \mathbf{Z}
$$
\n
$$
\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix} \in \mathbf{L}
$$
\n
$$
A^{2k} = 5^k \begin{bmatrix} F_{4k+1} & F_{4k} \\ F_{4k} & F_{4k-1} \end{bmatrix} \in \mathbf{F}
$$
\n
$$
A^{2k+1} = 5^k \begin{bmatrix} L_{4k+3} & L_{4k+2} \\ L_{4k+2} & L_{4k+1} \end{bmatrix} \in \mathbf{L}
$$

Examples

Ring Extension

 \circ Now we have a result for the Fibonacci and Lucas numbers, we would like to make a generalization for all recursive sequences, 1 2 , , . *A A A* **n n 10 m**
 n a result for the Film would like to make

sequences,
 $n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta$
 α would require in
 $\begin{bmatrix} \alpha b + \beta a & b \\ b & a \end{bmatrix}$. **SiON**
 a result for the Fibonacci and Lucas

would like to make a generalization for

equences,
 $= \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}$.

Would require in the 2x2 case is
 $\begin{bmatrix} \alpha b + \beta a & b \\ b & a \end{bmatrix}$.

tity is in Ω , th **1**

ult for the Fibonacci

l like to make a gen

nces,
 $\begin{aligned}\na + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}.\n\end{aligned}$ ld require in the 2x.
 $\begin{aligned}\nb + \beta a & b \\
b & a\n\end{aligned}$ is in Ω , then we mus It for the Fibonacci
like to make a gen
ces,
+ βA_{n-2}, α,β ∈ **Z**.
d require in the 2x.
+ βa b].
in Ω, then we mus
xtension. Thus, β **DN**
 Solution
 EXECUTE:

The same set of the Fibonacci and Lucas

we would like to make a generalization for

re sequences,
 $A_n = \alpha A_{n-1} + \beta A_{n-2}$, $\alpha, \beta \in \mathbb{Z}$.

we would require in the 2x2 case is
 $\begin{bmatrix} \alpha b + \beta a & b \\ b & a \end{bmatrix}$.
 n n sion
 n
 i e a result for the File
 i sequences,
 $A_n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta$

we would require in
 $\begin{bmatrix} \alpha b + \beta a & b \\ b & a \end{bmatrix}$.

entity is in Ω , then v

ne ring extension. T

we will concentrate c
 $n = \alpha$ = + − −**^Z**

$$
A_n = \alpha A_{n-1} + \beta A_{n-2}, \quad \alpha, \beta \in \mathbb{Z}.
$$

 \circ The matrix we would require in the 2x2 case is

$$
\begin{bmatrix} ab+\beta a & b \\ b & a \end{bmatrix}.
$$

- \circ Since the identity is in Ω, then we must have the identity in the ring extension. Thus, $\beta = 1$.
- Therefore, we will concentrate on

$$
A_n = \alpha A_{n-1} + A_{n-2}, \quad \alpha \in \mathbb{Z}.
$$

Ring Extension

$$
\sum_{a} \sum_{a} \begin{bmatrix} ab + a & b \\ b & a \end{bmatrix}
$$

More Definitions

 $\mathbf{\hat{W}}^{\bullet}$ this set as P_{w} . Definition: A Period-1 sequence is any sequence when expressed in matrix form will be closed under multiplication. Define

Define this set as $P_{\mathcal{D}}$. A Period-2 sequence is any sequence when expressed in matrix form will be closed under multiplication in the union of the Period-2 sequence and its complimentary Period-1 sequence. Definition:

Isomorphism

Isomorphism
\nTheorem:
$$
\Omega_{\alpha} = \mathbb{Z}[\sigma_{\alpha}] \cong \mathbb{Z}[\phi], \quad \sigma_{\alpha} = \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}
$$

\nTheorem: $(1 + \alpha \phi)^{2n} / \phi^2 - \alpha \phi - 1 = A_{2n-1} + A_{2n} \phi$
\nsuch that
\n $A_n = \alpha A_{n-1} + A_{n-2}, \quad A_1 = 1, A_2 = \alpha$.
\nExample:

Theorem:

$$
\mathbf{p} = \mathbf{p} \mathbf{p} \mathbf{p}
$$
\n
$$
\Omega_{\alpha} = \mathbf{Z}[\sigma_{\alpha}] \cong \mathbf{Z}[\phi], \quad \sigma_{\alpha} = \begin{bmatrix} \alpha & 1 \\ 1 & 0 \end{bmatrix}
$$
\n
$$
(1 + \alpha \phi)^{2n} / \phi^{2} - \alpha \phi - 1 = A_{2n-1} + A_{2n} \phi
$$
\nsuch that

\n
$$
A_{n} = \alpha A_{n-1} + A_{n-2}, \quad A_{1} = 1, A_{2} = \alpha.
$$
\n
$$
(1 + \phi)^{3} / \phi^{2} - \phi - 1 =
$$

$$
A_n = \alpha A_{n-1} + A_{n-2}, \quad A_1 = 1, A_2 = \alpha.
$$

Example:

$$
Pα = Z[σα] ≡ Z[φ], σα = \begin{bmatrix} α & 1 \\ 1 & 0 \end{bmatrix}
$$

\n(1+αφ)²ⁿ/φ² - αφ - 1 = A_{2n-1} + A_{2n}φ
\nsuch that
\nA_n = αA_{n-1} + A_{n-2}, A₁ = 1, A₂ = α.
\n(1+φ)³/φ² - φ - 1 =
\n(1+3φ + 3φ² + φ³)/φ² - φ - 1
\n= 5 + 8φ = F₅ + F₆φ

Period-2 2x2 Matrices

Theorem: We express a Period-1 sequence as

2 X2 Matrices
express a Period-1 sequence as
{...1,
$$
\alpha
$$
, $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...}.

$$
\Lambda_{\alpha} = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}.
$$
press a Period-2 sequence for odd α as

2 **X2 Matrices**

xpress a Period-1 sequence as

{...1, α , $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...}.
 $\Lambda_{\alpha} = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

ress a Period-2 sequence for odd α as

{... α , $\alpha^2 + 2$, $\alpha^3 + 3\alpha$...}.
 $\eta_{\alpha} = \begin$ 2x2 Matrices

ress a Period-1 sequence as

..1, α , $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...}.
 $\alpha = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

s a Period-2 sequence for odd α as

.. α , $\alpha^2 + 2$, $\alpha^3 + 3\alpha$...}.
 $\alpha = \begin{bmatrix} \alpha^3 + 3\alpha & \alpha^2 +$ 2 **Matrices**

Period-1 sequence as
 $(\alpha^2 + 1, \alpha^3 + 2\alpha...)$
 $(\alpha^2 + 1, \alpha)$
 $(\alpha - 1)^2$
 $(\alpha - 1)^2$
 $(\alpha - 1)^2$
 $(\alpha - 2)^2$
 $(\alpha - 1)^3 + 3\alpha$
 $(\alpha^2 + 2)^2$
 $(\alpha - 1)^2 + 2(\alpha - 1)^2$
 $(\alpha - 1)^2 + 2(\alpha - 1)^2$
 $(\alpha - 1)^2 + 2(\alpha - 1)^2$ **atrices**
 $\frac{1}{1}$ = $\frac{1}{1}$ = $\frac{1}{1}$
 $\frac{1}{1}$
 $\frac{1}{1}$
 $\frac{1}{2}$ **2X2 Matrices**

xpress a Period-1 sequence as

{...1, α , $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...}.
 $\Lambda_{\alpha} = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

ress a Period-2 sequence for odd α as

{... α , $\alpha^2 + 2$, $\alpha^3 + 3\alpha$...}.
 $\eta_{\alpha} = \begin{$ **2 Matrice**

Period-1 sequence
 $\begin{bmatrix} \n\alpha^2 + 1 & \alpha \\
\alpha^2 + 1 & \alpha \\
\alpha & 1\n\end{bmatrix}$.

riod-2 sequence for
 $\begin{bmatrix} 2 + 2, \alpha^3 + 3\alpha \dots \\ \alpha^2 + 2 & \alpha \end{bmatrix}$.
 $\begin{bmatrix} \n\alpha^3 + 3\alpha & \alpha^2 + 2 \\
\alpha^2 + 2 & \alpha \end{bmatrix}$. Matrices

Period-1 sequence as
 $,\alpha^2 + 1, \alpha^3 + 2\alpha..\ \alpha^2 + 1 \alpha$
 α 1]

iod-2 sequence for ode

+2, $\alpha^3 + 3\alpha...\}$.
 $\alpha^3 + 2\alpha$
 $\alpha^2 + 2\alpha$
 $\alpha^2 + 2\alpha$ $2 \sqrt{2}$ **2X2 Matrices**

spress a Period-1 sequence as

{... 1, α , $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...}.
 $\Lambda_{\alpha} = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

ess a Period-2 sequence for odd α a

{... α , $\alpha^2 + 2$, $\alpha^3 + 3\alpha$...}.
 $\eta_{\alpha} = \begin{b$ Matrices

od-1 sequence as
 $\begin{bmatrix}\n2 + 1, \alpha^3 + 2\alpha \dots\n\end{bmatrix}$
 $\begin{bmatrix}\n+1 & \alpha \\
\alpha & 1\n\end{bmatrix}$

-2 sequence for odd α as

2, $\alpha^3 + 3\alpha \dots$).

3 $\alpha \alpha^2 + 2$

-2 α

-2 sequence for even α as

+2, $\alpha^3 + 3\alpha \dots$ }. $\eta_{\alpha} = \begin{vmatrix} \alpha & 3\alpha & \alpha & 2 \\ \alpha^2 + 2 & \alpha & \alpha \end{vmatrix}.$ 2x2 Matrices

ress a Period-1 sequence as

..1, α , $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...}.
 $\alpha = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

ss a Period-2 sequence for odd α as

... α , $\alpha^2 + 2$, $\alpha^3 + 3\alpha$...}.
 $\alpha = \begin{bmatrix} \alpha^3 + 3\alpha & \alpha^2$ 2 Matrices

a Period-1 sequence as
 $\alpha, \alpha^2 + 1, \alpha^3 + 2\alpha...$ }.
 $\begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

eriod-2 sequence for odd α as
 $\alpha^2 + 2, \alpha^3 + 3\alpha...$ }.
 $\alpha^3 + 3\alpha \alpha^2 + 2$
 $\alpha^2 + 2 \alpha$

Period-2 sequence for even α as 2 Matrices

Period-1 sequence as
 $\kappa, \alpha^2 + 1, \alpha^3 + 2\alpha...$ }
 $\begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$
 $\begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$

riod-2 sequence for odd α as
 $\begin{bmatrix} \alpha^2 + 2, \alpha^3 + 3\alpha & \alpha^2 + 2 \\ \alpha^2 + 2, \alpha^3 + 3\alpha... \end{bmatrix}$

Per Matrices

eriod-1 sequence as
 $\alpha^2 + 1, \alpha^3 + 2\alpha...$ }.
 $\alpha^2 + 1 \alpha$
 α 1].

od-2 sequence for odd α as
 $+ 2, \alpha^3 + 3\alpha...$ }.
 $+ 3\alpha \alpha^2 + 2$
 $^2 + 2 \alpha$
 2
 2 + 2 α Matrices

riod-1 sequence as
 $\alpha^2 + 1, \alpha^3 + 2\alpha ...$
 $\alpha^2 + 1 \alpha$

dd-2 sequence for odd α
 α -2, $\alpha^3 + 3\alpha ...$ }
 $\alpha^2 + 2$
 $+2 \alpha$

iod-2 sequence for even
 $+2, \alpha^3 + 3\alpha ...$ }
 $\alpha^3 + 3\alpha$
 $\alpha^2 + 2$
 $\alpha^3 + 2\alpha$ Theorem: We express a Period-2 sequence for odd α as

X2 Matrices

SS a Period-1 sequence as
 $1, \alpha, \alpha^2 + 1, \alpha^3 + 2\alpha ...$
 $= \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

a Period-2 sequence for odd α as
 $\alpha, \alpha^2 + 2, \alpha^3 + 3\alpha ...$
 $= \begin{bmatrix} \alpha^3 + 3\alpha & \alpha^2 + 2 \\ \alpha^2 + 2 & \alpha \end{bmatrix}$.

SS a Period- $\eta_{\alpha} = \frac{1}{2} \left| \alpha^2 + 2 \alpha \right|$ $\frac{1}{2}$ {... α , α^2 + 2, α^3 + 3 α ...}. 2 2 **Matrices**

express a Period-1 sequence as

{...1, α , $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...}.
 $\Lambda_{\alpha} = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

ress a Period-2 sequence for odd α as

{... α , $\alpha^2 + 2$, $\alpha^3 + 3\alpha$...}.
 $\eta_{\alpha} = \begin$ **12 Matrices**
 13 a Period-1 sequence as
 $(\alpha, \alpha^2 + 1, \alpha^3 + 2\alpha...)$
 $= \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

Period-2 sequence for odd α as
 $(\alpha^2 + 2, \alpha^3 + 3\alpha...)$.
 $\begin{bmatrix} \alpha^3 + 3\alpha & \alpha^2 + 2 \\ \alpha^2 + 2 & \alpha \end{bmatrix}$.

a Period-2 sequ 2x2 Matrices

press a Period-1 sequence as

....1, α , $\alpha^2 + 1$, $\alpha^3 + 2\alpha$...).
 $\Lambda_{\alpha} = \begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

ess a Period-2 sequence for odd α as
 $\{\ldots \alpha, \alpha^2 + 2, \alpha^3 + 3\alpha \ldots\}$.
 $\eta_{\alpha} = \begin{bmatrix} \alpha^3 +$ Matrices

Period-1 sequence as
 $(\alpha^2 + 1, \alpha^3 + 2\alpha...)$
 $(\alpha^2 + 1 - \alpha)$
 $(\alpha - 1)^2$

riod-2 sequence for odd α as
 $(\alpha + 2, \alpha^3 + 3\alpha...)$
 $(\alpha + 2, \alpha^3 + 3\alpha)$
 $(\alpha^2 + 2, \alpha^3 + 3\alpha)$

Period-2 sequence for even α as
 $x^2 + 2, \alpha$ Matrices

veriod-1 sequence as
 $\alpha^2 + 1$, $\alpha^3 + 2\alpha ...$
 $\alpha^2 + 1$ a
 α 1

iod-2 sequence for odd α as
 $+ 2$, $\alpha^3 + 3\alpha ...$
 $\alpha^2 + 2$
 α
 $\alpha^2 + 2$ a
 $\alpha^3 + 3\alpha ...$
 $\alpha^3 + 3\alpha ...$
 $\alpha^3 + 3\alpha - \alpha^2 + 2$
 $\alpha^3 + 2\alpha -$ Matrices
 $\begin{bmatrix}\n\text{ind-1 sequence as } \\
\frac{c^2 + 1}{\alpha} + 1 & \frac{1}{\alpha} \\
\frac{1}{\alpha} & 1\n\end{bmatrix}$.
 $\begin{bmatrix}\n\frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} \\
\frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha}\n\end{bmatrix}$.
 $\begin{bmatrix}\n-3\alpha & \frac{a^2 + 2}{\alpha} \\
+2 & \alpha\n\end{bmatrix}$.
 $\begin{bmatrix}\n\frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1$ 2 Matrices

Period-1 sequence as
 $x, \alpha^2 + 1, \alpha^3 + 2\alpha...$ }.
 $\begin{bmatrix} \alpha^2 + 1 & \alpha \\ \alpha & 1 \end{bmatrix}$.

Period-2 sequence for odd α as
 $\begin{bmatrix} \alpha^3 + 3\alpha & \alpha^2 + 2 \\ \alpha^2 + 2 & \alpha \end{bmatrix}$.

Period-2 sequence for even α as
 $\alpha^2 + 2, \alpha^$ 2x2 Matrices

= $\frac{2x^2 + 1}{\alpha}, \frac{\alpha^2 + 1}{\alpha^3 + 2\alpha...}$

= $\left[\frac{\alpha^2 + 1}{\alpha}, \frac{\alpha^3}{1}\right]$

= $\left[\frac{\alpha^2 + 1}{\alpha}, \frac{\alpha^3}{1}\right]$

= $\left[\frac{\alpha^3 + 3\alpha}{\alpha^2 + 2}, \frac{\alpha^3 + 2\alpha...}{\alpha}\right]$

= $\left[\frac{\alpha^3 + 3\alpha}{\alpha^2 + 2}, \frac{\alpha^2 + 2}{\alpha}\right]$

= $\frac{1}{2}\left[\frac$ Theorem: We express a Period-2 sequence for even α as

Proof for odd α

 $A_n = \alpha A_{n-1} + A_{n-2}$
 $x^2 = \alpha x + 1.$ $n = \alpha A_{n-1} + A_{n-2}$
 $\kappa^2 = \alpha x + 1.$ $= \alpha A_{n-1} + A_{n-2}$
= $\alpha x + 1$. we define the characteristic polynomial, $x^2 = \alpha x + 1$ $A_n = \alpha A_{n-1} + A_{n-2}$
 $x^2 = \alpha x + 1.$ **dd a**

ecursive sequence in the form, $A_n = \alpha A_{n-1} + A_{n-2}$

the characteristic polynomial, $x^2 = \alpha x + 1$.
 $\frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}$.
 $\frac{\alpha + \sqrt{\alpha^2 + 4}}{2}$. c **Odd a**

n a recursive sequence in the form, $A_n = \alpha A_{n-1} + A_{n-2}$

efine the characteristic polynomial, $x^2 = \alpha x + 1$.
 $x = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}$.
 $\phi = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2}$.
 $\phi^2 = \alpha \phi + 1$ ord \overrightarrow{a}

ecursive sequence in the form, $A_n = \alpha A_{n-1} + A_{n-2}$

the characteristic polynomial, $x^2 = \alpha x + 1$.
 $\frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}$.
 $\frac{\alpha + \sqrt{\alpha^2 + 4}}{2}$.
 $= \alpha \phi + 1$
 $\frac{\alpha^2 + 2\phi^2}{2} = \sqrt{\alpha^2 + 4} (\alpha \phi + 1)$
 $\frac{\alpha^2 + 2$ 1 $d \alpha$

ursive sequence in the form, $A_n = \alpha A_{n-1} + A_{n-2}$

e characteristic polynomial, $x^2 = \alpha x + 1$.
 $\frac{\pm \sqrt{\alpha^2 + 4}}{2}$.
 $\frac{\alpha \phi + 1}{1 + 4\phi^2} = \sqrt{\alpha^2 + 4} (\alpha \phi + 1)$
 $\frac{2}{1 + 2}\phi + \alpha$

the characteristic polynomial plays Definition: Given a recursive sequence in the form, $A_n = \alpha A_{n-1} + A_{n-2}$

$$
x = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}.
$$

$$
\phi = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2}.
$$

Proof:

r odd **Q**
\nn a recursive sequence in the form,
$$
A_n = \alpha A_{n-1} + A_{n-2}
$$

\nefine the characteristic polynomial, $x^2 = \alpha x + 1$.
\n
$$
x = \frac{\alpha \pm \sqrt{\alpha^2 + 4}}{2}
$$
\n
$$
\phi = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2}
$$
\n
$$
\phi^2 = \alpha \phi + 1
$$
\n
$$
\sqrt{\alpha^2 + 4} \phi^2 = \sqrt{\alpha^2 + 4} (\alpha \phi + 1)
$$
\n
$$
= (\alpha^2 + 2)\phi + \alpha
$$
\nant of the characteristic polynomial plays an

Note: The discriminant of the characteristic polynomial plays an important role.

Proof for odd α

$$
\begin{aligned}\n\mathbf{M}_{a} &= \begin{bmatrix} \alpha^{3}+3\alpha & \alpha^{2}+2 \\ \alpha^{2}+2 & \alpha \end{bmatrix} \\
\eta_{a}^{2} &= \begin{bmatrix} \alpha^{6}+7\alpha^{4}+13\alpha^{2}+4 & \alpha^{5}+6\alpha^{3}+8\alpha \\ \alpha^{5}+6\alpha^{3}+8\alpha & \alpha^{4}+5\alpha^{2}+4 \end{bmatrix} \\
&= (\alpha^{2}+4) \begin{bmatrix} \alpha^{4}+3\alpha^{2}+1 & \alpha^{3}+2\alpha \\ \alpha^{3}+2\alpha & \alpha^{2}+1 \end{bmatrix} = (\alpha^{2}+4)\Lambda_{a}^{2} \\
\mathbf{M}_{a}^{2n} &= \Lambda_{a}^{n} \in \mathbf{P}_{k}\n\end{aligned}
$$

$\alpha = 5$

 $\Lambda_5 = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}.$ $\{...1,5,26,...\}.$ e:

od-1

26,...}.

26 5

5 1

5 1

5 0

5 2

5 2

5 2

5 2

7 =5

mple:

Period-1

1,5,26,...}.

= $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.

26 5 $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$
 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$

26 5 $\begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$ 5

ople:

eriod-1

5, 26,...}.
 $=\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 5 & 5 \\ 1 & 0 \end{bmatrix}$

135 26

26 5
 $\begin{bmatrix} 26 & 5 \\ 26 & 5 \end{bmatrix}$

11 0 1 0 1

135 29 5

ple:

riod-1

5, 26,...}.
 $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 5 \\ 1 & 0 \end{bmatrix}$

35 26

26 5
 $\begin{bmatrix} 5 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

701 135

35 26 = 5

umple:

Period-1

.1,5,26,...}.
 $=\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.

26 $5 \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$
 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$

26 $5 \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 70$ = 5

mple:

Period-1

1,5,26,...}.

= $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.

26 $5 \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$
 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$

5 $1 \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

701 135

135 26 5

pple:

eriod-1

5, 26,...}.
 $=\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.

5 $\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

135 26

26 5

5 $\begin{bmatrix} 5 \\ 26 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

701 135

135 26 5

mple:

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5, 26,...}.
 $=\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 5 & 5 \\ 1 & 0 \end{bmatrix}$

135 26

26 5
 $\begin{bmatrix} 26 & 5 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

701 135

135 26 Period - Example:

1=5

Example:

Period-1

Period-2
 $\{...1,5,26,...\}$.
 $\Lambda_s = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\eta_s = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 5 & 2 \end$ $\{...5, 27, 140...\}$. Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.

¹⁴⁰ 27 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$ [135 26] = 5

umple:

Period-1

1,5,26,...}.

= $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 135 & 1 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 135 & 1 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 135 & 1 \\ 26 & 5 \end{bmatrix}$
 =5

ample:

Period-1

.1,5,26,...}.
 $s = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 26 & 5 \\ 1 & 0 \end{bmatrix}$
 $\begin{bmatrix} 26 & 5 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 28 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 28 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix}$ =5

ample:

Period-1

..1,5,26,...}.
 $\begin{bmatrix}\n\text{Period-1} & \text{Period-2} \\
\text{1.1,5,26,...}\n\end{bmatrix}$.
 $\begin{bmatrix}\n\text{Period-2} & \text{1.5,27,140...}\n\end{bmatrix}$.
 $\begin{bmatrix}\n\text{140} & 27 \\
\text{27} & \text{5}\n\end{bmatrix}$.
 $\begin{bmatrix}\n26 & 5 \\
5 & 1\n\end{bmatrix}\n\begin{bmatrix}\n5 & 1 \\
26 & 5\n\end{bmatrix$ =5

ample:

Period-1

.1,5, 26,...}.
 $\begin{aligned}\n\mathbf{r}_5 &= \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}.\n\end{aligned}$
 $\mathbf{r}_8 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}.\n\end{aligned}$
 $\mathbf{r}_9 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}.\n\end{aligned}$
 $\mathbf{r}_1 = \begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 \math = 5

ample:

Period-1

..1,5,26,...}.
 $S = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 26 & 5 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix}$ = 5

ample:

Period-1

..1,5,26,...}.
 $S = \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}$.
 $\begin{bmatrix} 26 & 5 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 5 & 1 \\ 2 & 6 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix} \begin{bmatrix}$ =5

ample:

Period-1

.1,5, 26,...}.
 $\begin{aligned}\n\mathbf{r}_5 &= \begin{bmatrix} 26 & 5 \\ 5 & 1 \end{bmatrix}.\n\end{aligned}$
 $\mathbf{r}_8 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}.\n\end{aligned}$
 $\mathbf{r}_9 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}.\n\end{aligned}$
 $\mathbf{r}_1 = \begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 \math 1 Period-2 Period-2

{...5, 27, 140...}.
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.

140 27 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$

140 27 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

[2775 277] Period-2

...5, 27, 140...}
 $\eta_s = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$
 $27 \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$
 $29 \begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $29 \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$
 $29 \begin{bmatrix} 3775 & 727 \\ 727 & 140 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 & 0 & 27 \\ 27 & 14 & 4 & 5 \\ 27 & 5 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$
 $\begin{bmatrix} 27 & 5 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 27 & 5 \\ 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$
 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$
 $=29\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$ od-2

27,140...}
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$
 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

35 26

26 5

27 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$ 27 5

5 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$ 27 5

3775 727

727 140 Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$

140 27 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$

140 27 $\begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 3775 & 727 \\ 7$ Period-2

...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$
 $27 \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$
 $29 \begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $29 \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$
 $29 \begin{bmatrix} 3775 & 727 \\ 727 & 140 \end{bmatrix}$ iod-2

27,140...}
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$
 $\begin{bmatrix} 27 \\ 5 \\ 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 27 \\ 27 \\ 5 \\ 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

3775 727

727 140 $=29\begin{bmatrix} 3775 & 727 \\ 727 & 140 \end{bmatrix}$ od-2

27,140...}

[140 27]

₂₇ 5]

27 [27 5]

5 [5 2]

35 26]

26 5]

27 [27 5]

5 [5 2]

5 27 [27 5]

5 27 [27 5]

5 27 [27 5]

5 27 [27 5] Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 &$ Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 26 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 &$ Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 &$ Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 &$ Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 &$ Period-2

{...5, 27, 140...}
 $\eta_5 = \begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix}$.
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix}$

= 29 $\begin{bmatrix} 135 & 26 \\ 26 & 5 \end{bmatrix}$
 $\begin{bmatrix} 140 & 27 \\ 27 & 5 \end{bmatrix} \begin{bmatrix} 27 & 5 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 27 &$

Generalized Recursive Relations Relations
 Vaez.
 Z.

1 \mathbf{I}_n $1 \begin{array}{c|c} \n\hline\n\end{array}$ $1 \quad \Box$ Theorem: $\sigma_{n}^{n} = \begin{bmatrix} A_{n+1} & A_{n} \end{bmatrix}$ $1 \quad \Box$ *n n n* **d Recurs**
 $\begin{bmatrix} A_1 \\ A_0 \end{bmatrix}$ is the sl
 F_n
 F_{n-1}
 $\begin{bmatrix} F_n \\ F_n \\ H_n \\ A_{n-1} \end{bmatrix}$ **ized Recurs**
 $\overline{R} = \begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the sh
 $\begin{bmatrix} n \\ n \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
 $\begin{bmatrix} n \\ n \\ n \end{bmatrix} = \begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{bmatrix}$ **d** Recurs
 $\begin{bmatrix} A_1 \\ A_0 \end{bmatrix}$ is the sl
 F_n
 F_{n-1}
 \vdots
 F_{n-1}
 \vdots
 F_{n-1}
 A_{n-1}
 \vdots
 A_{n+1}
 A_n **ed Recursive R**
 $\begin{bmatrix} A_2 & A_1 \ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha$
 $\begin{bmatrix} F_{n+1} & F_n \ F_n & F_{n-1} \end{bmatrix}$
 $\begin{bmatrix} A_{n+1} & A_n \ A_n & A_{n-1} \end{bmatrix}$ **PO Recursive F**
 $\begin{bmatrix} A_2 & A_1 \ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha$
 $\begin{bmatrix} F_{n+1} & F_n \ F_n & F_{n-1} \end{bmatrix}$
 $\begin{bmatrix} A_{n+1} & A_n \ A_n & A_{n-1} \end{bmatrix}$ **PO Recursive R**
 $\begin{bmatrix} A_2 & A_1 \ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha$
 $\begin{bmatrix} F_{n+1} & F_n \ F_n & F_{n-1} \end{bmatrix}$
 $\begin{bmatrix} A_{n+1} & A_n \ A_n & A_{n-1} \end{bmatrix}$
 $\begin{bmatrix} A_{n-1}A_m + A_nA_{m+1} \end{bmatrix}$ **ilized Recursive F**
 $\sigma_{\alpha} = \begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha$
 $\sigma_1^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
 $\sigma_{\alpha}^n = \begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{bmatrix}$
 $\lambda_{n+m} = A_{n-1}A_m + A_nA_{m+1}$
 $\sigma^m \sigma^{n-1} = \sigma^{m+n-1}$. $+1$ $\binom{n}{n}$ -1 \Box −1 <u>|</u> **zed Recursive Relations**
 $=\begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $=\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
 $=\begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{bmatrix}$ zed Recursive Relations
 $=\begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $=\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
 $=\begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{bmatrix}$
 $= A_{n-1}A_m + A_nA_{m+1}$
 \vdots \vdots \vdots \vdots \vdots \vdots \vdots alized Recursive Rela $\sigma_{\alpha} = \begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $\sigma_{\alpha}^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
 $\sigma_{\alpha}^n = \begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{bmatrix}$
 $A_{n+m} = A_{n-1}A_m + A_nA_{m+1}$
 $\sigma_{\alpha}^m \sigma_{\alpha$ *n*_a = $\begin{bmatrix} A_2 & A_1 \ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $\sigma_i^{\alpha} = \begin{bmatrix} F_{n1} & F_n \ F_n & F_{n-1} \end{bmatrix}$
 $\sigma_{\alpha}^{\alpha} = \begin{bmatrix} A_{n+1} & A_n \ A_n & A_{n+1} \end{bmatrix}$
 $\sigma_m^{m} \sigma_{\alpha}^{n-1} = \sigma_{\alpha}^{m+n-1}$.
 $\sigma_{\alpha}^{m} \sigma_{\alpha}^{n-1} = \sigma_{$ $\begin{bmatrix} A_1 \\ A_0 \end{bmatrix}$ is the shift map
 $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$
 $\begin{bmatrix} A_n \\ A_{n-1} \end{bmatrix}$
 $\begin{bmatrix} A_m + A_n A_{m+1} \\ A_1 = \sigma_{\alpha}^{m+n-1} \end{bmatrix}$
 $\begin{bmatrix} 1 \\ A_{2k} A_{2k-1} - A_{2k}^2 = 1 \end{bmatrix}$ Proof: $\sigma_{\alpha}^{m} \sigma_{\alpha}^{n-1} = \sigma_{\alpha}^{m+n-1}$. ized Recursive Relations
 $=\begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $\begin{bmatrix} x \\ x \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
 $\begin{bmatrix} x \\ x \\ x \end{bmatrix} = \begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n+1} \end{bmatrix}$
 $\begin{bmatrix} m \\ m \end{bmatrix} = A_{n-1}A_m + A$ $\begin{bmatrix} A_1 \\ A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $+1$ F_n
 F_{n-1}
 $+1$ A_n
 $+1$ A_n
 $+1$
 A_{n-1}
 $+1$
 A_{n-1}
 $+1$
 lized Recursive
 $\begin{aligned}\n\tau_{\alpha} &= \begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\sigma_1^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \\
\sigma_{\alpha}^n &= \begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n-1} \end{bmatrix} \\
\sigma_m^m \sigma_{\alpha}^{n-1} &= \sigma_{\alpha}^{m+n-1}.\n\end{aligned}$ $\sigma_{\alpha}^m \sigma_{\alpha}^{n-1} = \sigma_{$ 2 1 2 2 1 2 ¹ *A A A A k k k k* − − + − = **c** $\overline{D}_\alpha = \begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $\sigma_i^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$
 $\sigma_\alpha^n = \begin{bmatrix} A_{n+1} & A_n \\ A_n & A_{n+1} \end{bmatrix}$
 $\sigma_\alpha^m \sigma_\alpha^{n-1} = \sigma_\alpha^{m+n-1}$.
 $\sigma_\alpha^m \sigma_\alpha^{n-1} = \sigma_\$ Theorem: Definition: $\frac{d \text{ Recur}}{A_1 \left(A_0\right)}$ is the $\frac{d\text{ Recur}}{\left(\frac{2}{\pi} - A_1\right)}$ is the
 $\left[\frac{1}{n+1} - F_n\right]$ $\begin{bmatrix} A_2 & A_1 \ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall A_1$
 F_{n+1} F_n F_n **and Recursive 1
** *A***₂** *A*₁ *A*₀ is the shift map \forall ₁ *F*_{*n*+1} *F*_{*n*} *F*_{*n*-1} *A*¹ $\sigma_{\alpha} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the shift map ed Recursive Relations
 $\begin{bmatrix} A_2 & A_1 \\ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $\begin{bmatrix} F_{n+1} & F_n \end{bmatrix}$ **zed Recursive Relations**
 $=\begin{bmatrix} A_2 & A_1 \ A_1 & A_0 \end{bmatrix}$ is the shift map $\forall \alpha \in \mathbb{Z}$.
 $=\begin{bmatrix} F_{n+1} & F_n \ F_n & F_{n-1} \end{bmatrix}$ Theorem: $A_{n+m} = A_{n-1}A_m + A_nA_{m+1}$ Theorem: $A_{2k-1}^2 + \alpha A_{2k} A_{2k-1} - A_{2k}^2 = 1$ Proof: $\det(\Lambda^n_{\alpha} \Lambda_{\alpha}) = 1$.

The Shift Map

Example: $\alpha = 2$.

$$
\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}
$$

 $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ Per 2 1]

1 0]

Per

{...0,1,3 $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$
Period-2 The Shift Map

Example: $\alpha = 2$.
 $\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Period-1

{...0,1,2,5,12...}

{...0,1,3,7,17...} Period-1 Period-2 mple: $\alpha = 2$.
 $\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Period-1

1,2,5,12...} {...0,1,3,7,17...}
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$ veriod-1
 $\begin{bmatrix} 1, 2, 5, 12... \end{bmatrix}$
 $\begin{bmatrix} 2 & 1 \end{bmatrix}$
 $\begin{bmatrix} 12 & 5 \end{bmatrix}$ mple: $\alpha = 2$.
 $\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Period-1

, 1, 2, 5, 12...}
 $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$ mple. $\alpha = 2$.
 $\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Period-1

1, 1, 2, 5, 12...}
 $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$
 $\begin{bmatrix} 2 & 1 \end{bmatrix}$ [12 5] riod-2

3, 7, 17...}

7 3

3 1 riod-2

3,7,17...}

7 3

3 1

9 1

1 [17 7] eriod-2
 $\begin{bmatrix} 3 & 7 \\ 3 & 1 \end{bmatrix}$
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$ eriod-2
 $\begin{bmatrix} 3 & 7 & 17 \ 3 & 1 \end{bmatrix}$
 $\begin{bmatrix} 7 & 3 \ 3 & 1 \end{bmatrix}$ The Shift Map

Example: $\alpha = 2$.
 $\sigma_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Period-1
 $\begin{bmatrix} .0, 1, 2, 5, 12... \end{bmatrix}$
 $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 5 & 2 \end{bmatrix}$
 $\begin{bmatrix} .0, 1, 2, 5, 12... \end{bmatrix}$
 σ_{α}^{n} The Shift Map

Example: $\alpha = 2$.
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Period-1

{...0,1,2,5,12...}
 $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$
 $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 5 & 2 \end{bmatrix}$

{...0,1,2,5,12...}
 σ_{α}^{n} $\sigma_{R_{\alpha}}^{n}$ The Shift Map

Example: $\alpha = 2$.
 $\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Period-1
 $\{\ldots, 0, 1, 2, 5, 12\ldots\}$
 $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$
 $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 5 \\ 5 & 2 \end{bmatrix}$
 $\begin{bmatrix} 7 & 3 \\$ The Shift Map

Example: $\alpha = 2$.
 $\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Period-1
 $\begin{bmatrix} 1.0,1,2,5,12... \end{bmatrix}$
 $\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$
 $\begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$
 $\begin{bmatrix}$ Period-2

...0, 1, 3, 7, 17...}
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$

7 $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 7 \\ 7 & 3 \end{bmatrix}$

{...0, 1, 3, 7, 17...} Period-2
{...0, 1, 3, 7, 17...}
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 7 \\ 7 & 3 \end{bmatrix}$
{...0, 1, 3, 7, 17...} **apple 10**
 $\sigma_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$

Period-2
 $\{\ldots 0, 1, 3, 7, 17 \ldots\}$
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix}$
 $\begin{bmatrix} 7 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 17 & 7 \\ 7 & 3 \end{bmatrix}$
 $\{\ldots 0, 1, 3, 7, 17 \ldots\}$
 \mathbf{P}_{\math σ^n_α σ^n_α

A Characteristic of Period-1 Matrices

 \circ Period-1 Matrices act as units to their correlating Period-2 Matrices

Theorem: Let $\delta \in E \subseteq \Omega_a$, then $\delta \in$

racteristic of Period-1 Matrices
 l Matrices act as units to their correlating
 let $\delta \in E \subseteq \Omega_a$, then $\delta \sigma_a^n \in E \ \forall \alpha \in \mathbb{Z}$
 $\begin{aligned}\ni & = \begin{bmatrix} ab+a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix} \\
i\sigma_a &= \begin{bmatrix} a^2b + a a + b & a b + a \\ ab +$ of Period

as units to the

then $\delta \sigma_{\alpha}^{n} \in E$
 $\begin{bmatrix} a & b+a \\ b & b \end{bmatrix} = \begin{bmatrix} B_4 & B_3 \end{bmatrix}$
 $\begin{bmatrix} a & b+a \\ b & a \end{bmatrix} = \begin{bmatrix} B_4 & B_5 \end{bmatrix}$ of Period

as units to the

then $\delta \sigma_{\alpha}^{n} \in E$
 $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$
 $\begin{bmatrix} a b + a \\ b \end{bmatrix} = \begin{bmatrix} B_4 \\ B_3 \end{bmatrix}$ $2h + \alpha a + h$ $\begin{bmatrix}\n\text{odd-1 Mat} \\
\text{their correl}\n\end{bmatrix}$ $\begin{aligned} \text{3:} \quad \mathbf{a} & = \mathbf{b} \ \mathbf{c} & = \mathbf{b} \ \mathbf{c} & = \mathbf{c} \ \mathbf{c} & = \mathbf{$ ristic of Pe
 z and a set as units
 $E \subseteq \Omega_a$, then δa
 $\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $\begin{bmatrix} a+b & ab+a & b \\ ab+a & b & b \\ B_3 & B_{n+2} & b \\ B_{n+1} & B_{n+1} \end{bmatrix}$. ristic of Pe
 z es act as uni
 $E \subseteq \Omega_a$, then δe
 $\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $\begin{bmatrix} a+b & ab+a & b \\ ab+a & b & b \\ B_3 & B_{n+2} & b \\ B_{n+1} & B_{n+1} \end{bmatrix}$. acteristic of Pe

Matrices

et $\delta \in E \subseteq \Omega_a$, then $\delta \sigma$
 $\begin{bmatrix}\nab+a & b \\
b & a\n\end{bmatrix} = \begin{bmatrix}\nB_3 & B_2 \\
B_2 & B_1\n\end{bmatrix}$
 $\begin{bmatrix}\na^b + a & b \\
b & a\n\end{bmatrix} = \begin{bmatrix}\nB_3 & B_2 \\
B_2 & B_1\n\end{bmatrix}$
 $\begin{bmatrix}\na^b + a & b \\
a^b + a & b\n\end{bmatrix}$
 $\begin{bmatrix}\na^b \\
B_{n+2}$ **eristic of F**

ices act as ur

ices
 $E \subseteq \Omega_a$, then d
 $\begin{bmatrix} a & b \\ a & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $\begin{bmatrix} a+b & ab+a & b \\ ab+a & b & M \\ B_{n+2} & B_{n+1} \end{bmatrix}$.
 σ_a^n σ_a^n σ_B *b a b B B* **b** and the set of Period-1 Mannum and the set of \vec{b} and \vec **c** and the set of Period-1 Matrices

ces act as units to their correlating

ces
 $E \subseteq \Omega_a$, then $\delta \sigma_a^a \in E \ \forall \alpha \in \mathbb{Z}$
 $\begin{bmatrix} a & b \\ a \\ a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $\begin{bmatrix} b+a a+b & a b+a \\ b+a a & b \end{bmatrix} = \begin{bmatrix} B_4 & B_3 \\ B_3 &$ **b** a b a b a b and their correlating
 b a b \sum_{a} *b* \sum_{a} *b* \sum_{b} *B* \sum_{a} *B* \sum_{b} *B* \sum_{b} *B* \sum_{b+a} *b* \sum_{b} \sum_{b+a} *B* \sum_{b+a} *B* \sum_{b+a} *B* \sum_{b+a} *B* \sum_{b+1} *B* \sum_{b+1} *B* **eristic of Period**

rices act as units to th

rices
 $\in E \subseteq \Omega_a$, then $\delta \sigma_a^n \in E$ \forall
 $\neq a$
 $\downarrow b$
 $\downarrow
 \downarrow **Bristic of Period-**

Frices act as units to therefore $\in E \subseteq \Omega_a$, then $\delta \sigma_a^n \in E \forall$
 $\sigma^+ a \quad b \quad a$ = $\begin{bmatrix} B_3 & B_2 \ B_2 & B_1 \end{bmatrix}$
 $\alpha^2 b + \alpha a + b \quad \alpha b + a$ = $\begin{bmatrix} B_4 & B_4 \ B_3 & B_{n+2} \end{bmatrix}$.
 $\begin{bmatrix} B_{n+3} & B_{n+2} \ B_{n+2} & B$ α , and α $\alpha b + a$ b b b , b , c $\delta =$ $|$ racteristic of Period-1 Matrices

1 Matrices

1 Matrices

2 Matrices

Let $\delta \in E \subseteq \Omega_a$, then $\delta \sigma_a^n \in E \ \forall \alpha \in \mathbb{Z}$
 $\delta = \begin{bmatrix} ab+a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $\delta \sigma_a = \begin{bmatrix} a^2b + aa + b & ab + a \\ ab + a & b \end{bmatrix} = \begin{bmatrix} B_a &$ racteristic of Period-1 Matrices

1 Matrices act as units to their correlating

2 Matrices

Let $\delta \in E \subseteq \Omega_a$, then $\delta \sigma_a^n \in E \ \forall \alpha \in \mathbb{Z}$
 $\delta = \begin{bmatrix} ab + a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_3 \\ B_2 & B_3 \end{bmatrix}$
 $\delta \sigma_a = \begin{bmatrix} a^2b + aa + b & ab +$ ristic of Period-1 Matrices

ces act as units to their correlating

ces
 $E \subseteq \Omega_a$, then $\delta \sigma_a^n \in E \ \forall \alpha \in \mathbb{Z}$
 $\left.\begin{bmatrix} a & b \\ a \\ a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix} \right]$
 $\left.\begin{bmatrix} b+aa+b & ab+a \\ b & b \end{bmatrix} = \begin{bmatrix} B_4 & B_3 \\ B_3 & B_2 \end{$ acteristic of Period-1 Matrices

Matrices

Matrices
 Let $\delta \in E \subseteq \Omega_a$, then $\delta \sigma_a^{\alpha} \in E \ \forall \alpha \in \mathbb{Z}$
 $\begin{bmatrix} ab + a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $\sigma_a = \begin{bmatrix} a^2b + a a + b & ab + a \\ ab + a & b \end{bmatrix} = \begin{bmatrix} B_4 & B_3 \\ B_3 & B_2 \end{$ cteristic of Period-1 Matrices

latrices act as units to their correlating

latrices
 $\delta \in E \subseteq \Omega_a$, then $\delta \sigma_a^a \in E \ \forall \alpha \in \mathbb{Z}$
 $\begin{bmatrix} ab + a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $= \begin{bmatrix} a^2b + \alpha a + b & \alpha b + a \\ \alpha b + a & b \end{b$ cteristic of Period-1 Matrices

latrices act as units to their correlating

latrices
 $\delta \in E \subseteq \Omega_a$, then $\delta \sigma_a^a \in E \ \forall \alpha \in \mathbb{Z}$
 $\begin{bmatrix} ab + a & b \\ b & a \end{bmatrix} = \begin{bmatrix} B_3 & B_2 \\ B_2 & B_1 \end{bmatrix}$
 $= \begin{bmatrix} a^2b + \alpha a + b & \alpha b + a \\ \alpha b + a & b \end{b$ Proof:

$$
\mathbf{M}%
$$

$$
\delta \sigma_{\alpha}^{n} = \begin{bmatrix} B_{n+3} & B_{n+2} \\ B_{n+2} & B_{n+1} \end{bmatrix}.
$$

Powers of Λ_{α} and η_{α}

Powers of
$$
\Lambda_{\alpha}
$$
 and η_{α}
\nTheorem: $\Lambda_{\alpha}^{n} = \begin{bmatrix} A_{2n+1} & A_{2n} \\ A_{2n} & A_{2n-1} \end{bmatrix}$
\nTheorem: For a even $\eta_{\alpha}^{2n-1} = \left(1 + \frac{\alpha^{2}}{4} \right)^{n-1} \eta_{\alpha} \Lambda_{\alpha}^{2n-2}$
\n $\eta_{\alpha}^{2n} = \left(1 + \frac{\alpha^{2}}{4} \right)^{n} \Lambda_{\alpha}^{2n}$
\nTheorem: For a odd $\eta_{\alpha}^{2n-1} = \left(\alpha^{2} + 4 \right)^{n-1} \eta_{\alpha} \Lambda_{\alpha}^{2n-2}$
\n $\eta_{\alpha}^{2n} = \left(\alpha^{2} + 4 \right)^{n} \Lambda_{\alpha}^{2n}$

Periodicity

Periodicity	
\n <p>Theorem:</p> \n $If \ C^{2} \in \Lambda_{\alpha}^{n}, then \ C \in \Lambda_{\alpha}^{n}, or \ C \in \eta_{\alpha}^{n}$ \n $If \ b = 0,$ \n $C = \begin{bmatrix} \alpha b + a & b \\ b & a \end{bmatrix}$ \n $C^{2} = \begin{bmatrix} (\alpha b + a)^{2} + b^{2} & \alpha b + 2ab \\ \alpha b + 2ab & b^{2} + a^{2} \end{bmatrix}$ \n $If \ b = \frac{-2a}{\alpha},$ \n $X = (\alpha b + a)^{2} + b^{2}$ \n $Y = \alpha b + 2ab$ \n $Z = b^{2} + a^{2}$ \n $X > 0, Z > 0,$ \n $let \ \alpha b + 2ab = 0$ \n $b(\alpha b + 2a) = 0$ \n $b = 0, \ b = \frac{-2a}{\alpha}$ \n	\n <p>Theorem:</p> \n $If \ b = 0,$ \n $If \ b = \frac{-2a}{\alpha},$ \n $C = -a \begin{bmatrix} \alpha^{2} + 3 & \frac{\alpha^{2} + 2}{\alpha} \\ \frac{\alpha^{2} + 2}{\alpha} & 1 \end{bmatrix}$ \n $C = -a \begin{bmatrix} \alpha^{2} + 3 & \frac{\alpha^{2} + 2}{\alpha} \\ \frac{\alpha^{2} + 2}{\alpha} & 1 \end{bmatrix}$ \n $a = k\alpha \longrightarrow \eta_{\alpha}, \alpha \text{ odd}$ \n $a = k\frac{\alpha}{2} \longrightarrow \eta_{\alpha}, \alpha \text{ even}$ \n

2 \bigcap

 $2 \mid$

1

4x4 Period-1 and Period-2 Matrices

Example:

$$
\begin{bmatrix}\n13 & 8 & 5 & 3 \\
8 & 5 & 3 & 2 \\
5 & 3 & 2 & 1 \\
3 & 2 & 1 & 1\n\end{bmatrix} = \begin{bmatrix}\nF_7 & F_6 & F_5 & F_4 \\
F_6 & F_5 & F_4 & F_3 \\
F_5 & F_4 & F_3 & F_2\n\end{bmatrix} = \begin{bmatrix}\n\Lambda_{\alpha}^3 & \Lambda_{\alpha}^2 \\
\Lambda_{\alpha}^2 & \Lambda_{\alpha}^1\n\end{bmatrix}
$$
\n
$$
B_1 = \begin{bmatrix}\n29 & 18 & 11 & 7 \\
18 & 11 & 7 & 4 \\
11 & 7 & 4 & 3 \\
7 & 4 & 3 & 1\n\end{bmatrix} = \begin{bmatrix}\nL_7 & L_6 & L_5 & L_4 \\
L_6 & L_5 & L_4 & L_3 \\
L_5 & L_4 & L_3 & L_2\n\end{bmatrix} = \begin{bmatrix}\n\eta_{\alpha} \Lambda_{\alpha}^2 & \eta_{\alpha} \Lambda_{\alpha}^1 \\
\eta_{\alpha} \Lambda_{\alpha}^1 & \eta_{\alpha} \Lambda_{\alpha}^0\n\end{bmatrix}
$$
\n
$$
B_1^2 = (5)(3)\begin{bmatrix}\n89 & 55 & 34 & 21 \\
55 & 34 & 21 & 13 \\
34 & 21 & 13 & 8 \\
21 & 13 & 8 & 5\n\end{bmatrix} = (5)(3)\begin{bmatrix}\n\Lambda_{\alpha}^5 & \Lambda_{\alpha}^4 \\
\Lambda_{\alpha}^4 & \Lambda_{\alpha}^3\n\end{bmatrix}
$$

Note: Similar to Period-1 and Period-2 2x2 matrices, it is possible to create a general formula for every α.

Higher Degree Periods

PEREFEREMALE PEREFEREM

For $n = 1, 2 \exists a, b, c \in \mathbb{Z}$ such that the primitive case of
 $(a + b\phi)^n = c\phi^n$ is true.

the proof is dependent on the fact the degree of the ring extension Theorem: For $n = 1, 2 \exists a, b, c \in \mathbb{Z}$ such that the primitive case of $(a+b\phi)^n = c\phi^n$ is true.

her Degree Periods
 For n = 1,2 $\exists a, b, c \in \mathbb{Z}$ such that the primitive case of $(a+b\phi)^n = c\phi^n$ is true.

The proof is dependent on the fact the degree of the ring extension $a \ge 2$. Proof: The proof is dependent on the fact the degree of the ring extension is 2.

Thus, there fails to exist periods of degree greater than 2.

The NxM case

 $P_{\psi} \cup P_{\vartheta}$. Definition: A complete orbit is an orbit closed under $P_{\psi} \cup P_{\vartheta}$.

Example:

Case
\n
$$
C_1 = \begin{bmatrix}\n11 & 7 \\
7 & 4 \\
4 & 3\n\end{bmatrix} \in L
$$
\n
$$
C_1C_1^T = 5 \begin{bmatrix}\n34 & 21 & 13 \\
21 & 13 & 8 \\
13 & 8 & 5\n\end{bmatrix} \in F
$$
\n
$$
C_1C_1^T C_1 = 5 \begin{bmatrix}\n573 & 361 \\
354 & 223 \\
219 & 138\n\end{bmatrix} \notin F, L
$$

This complete one orbit, not two; this fails to be a complete orbit.

Resulting Generalizations

Example 3
EXECUTE: EXECUTE: P
EXECUTE: *n* $\in \mathbf{Z}$, $m \in \mathbf{Z}$.
P
EXECUTE: P
EXECUTE: P
EXECUTE: P
EXECUTE: P
EXECUTE: P
EXECUTE: P
EXECUTE: P
EXECUTE: P
EXECUTE: P
E Theorem: We are guaranteed a complete orbit when we are given nxm matrix where $n \in \mathbb{Z}$, $m \in 2\mathbb{Z}$.

Theorem: Every nxn recursive matrix, M_n , where n is even, forms a ring.

Theorem: The set of nxm matrices that form a complete orbit is a semigroup.

Theorem: If $C^2 \in \mathbf{P}$, then

Problems of Interest

 Relations between Period-1 and Period-2 Sequences: **n ferrolling the SM**
 n Period-1 and Period
 $2^k = \gamma^k (A_1 + A_2 \phi)^{k(n+1)}.$
 norphisms

on maps
 s
 n maps

$$
(B_n + B_{n+1}\phi)^{2k} = \gamma^k (A_1 + A_2\phi)^{k(n+1)}
$$

- Finding more isomorphisms
	- Continued fraction maps
	- ⚫ Eigenvalue maps
	- ⚫ Determinant maps
- \circ Forming relationships for any power of every nxm recursive matrix in the Period-1 and Period-2 sets. 1 1 2 () () . *k k k n B B A A* **S Of Interest**

petween Period-1 and Period-2
 i:
 $\int_{n}^{1} + B_{n+1}\phi^{2k} = \gamma^{k}(A_{1} + A_{2}\phi)^{k(n+1)}$.

pre isomorphisms

ed fraction maps

lue maps

lue maps

lationships for any power of every nxm

matrix in the Period-1
- \circ Studying recursive relations of greater order.

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