# Diophantine Equations with Constraints

"Click and Clack's Clock"

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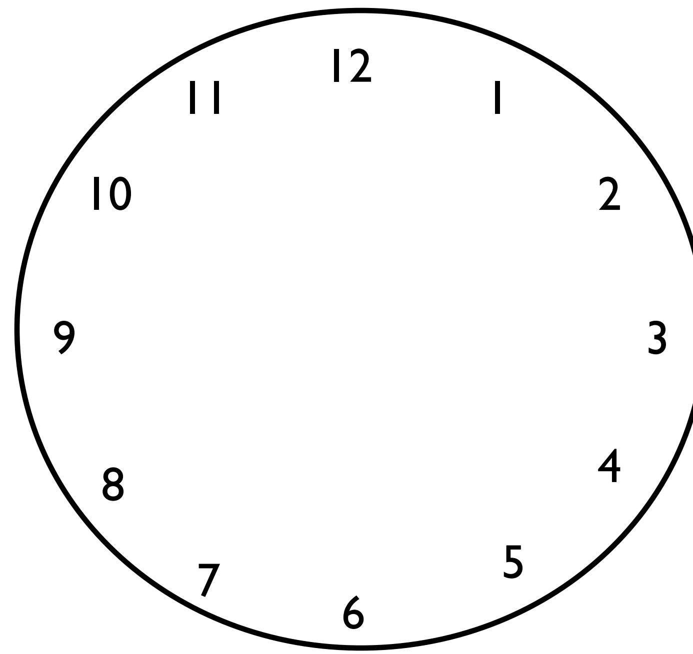
# Click and Clack's Clock

- Click and Clack are the hosts of an automotive repair show called Car Talk on National Public Radio. Each week, Click and Clack pose a brainteaser to their listeners and those listeners who submit correct answers to the problem have a chance of winning a prize.
- The following problem, titled "Dividing Time" was introduced on August 15th, 2005.

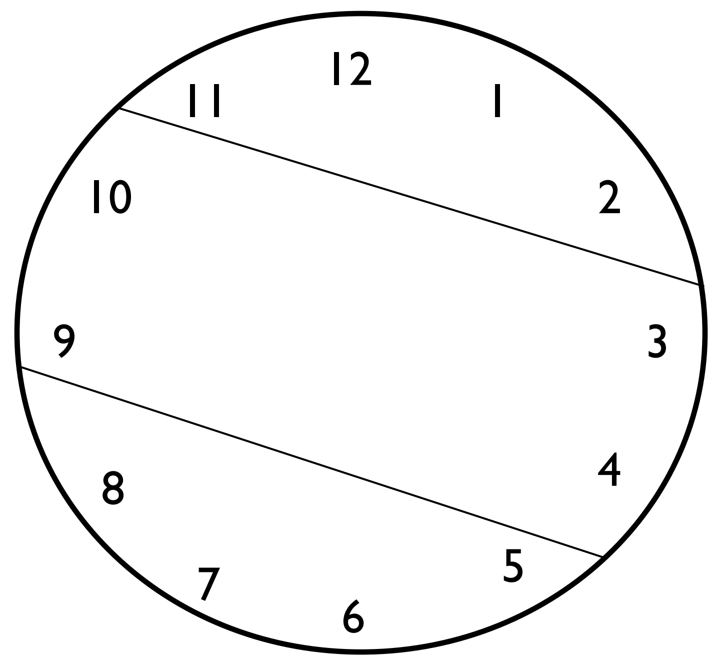
# The Problem

- Given a normal, 12-faced clock, how may a person make two cuts in the clock so that the numbers in each segment sum to the same number?



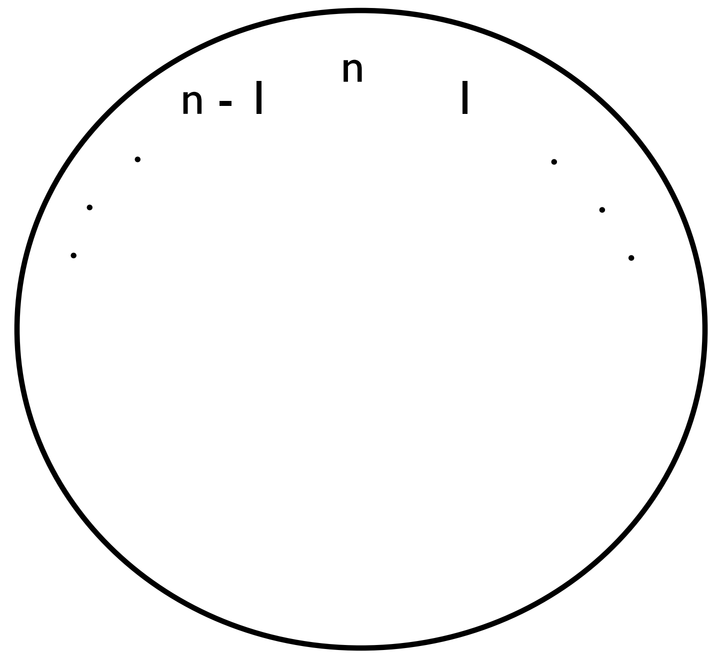
# The Solution

- 11 + 12 + 1 + 2 = 26
- 9 + 10 + 3 + 4 = 26
- 8 + 7 + 6 + 5 = 26



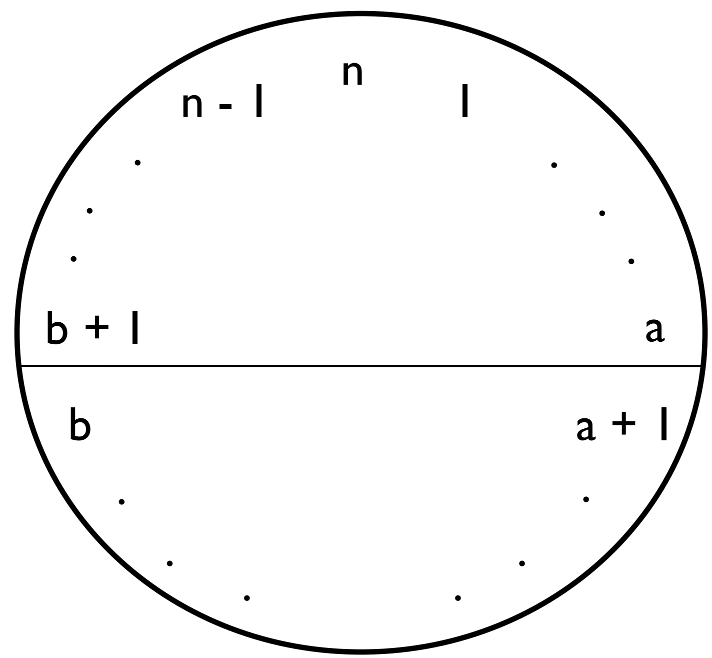
# What about an n-faced clock?

- So we want to generalize the problem and determine whether or not the problem is solvable for an n-faced clock.



# The One Cut Case

- Rather than doing the two-cut case, let's first back up and consider the case where only one cut may be made.



For any n:

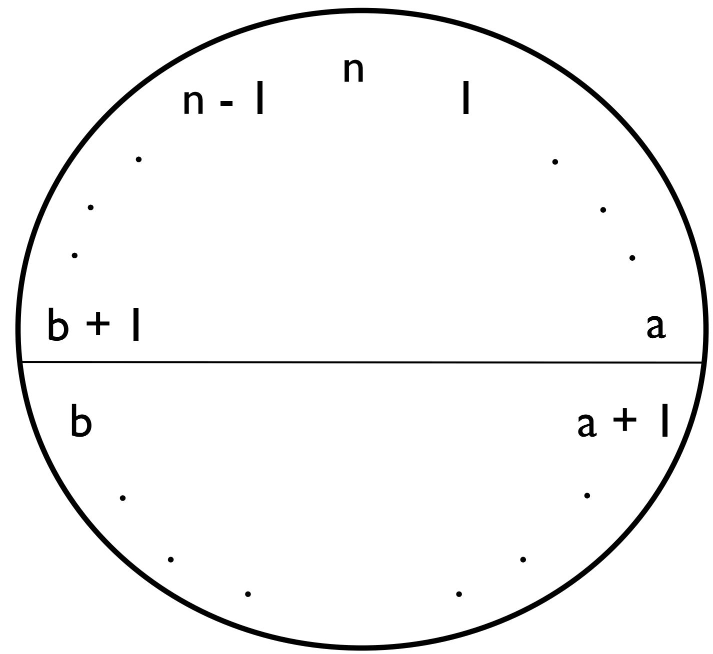
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

So for the one-cut case when only two segments are formed, we want each segment to contain half of this summation, or:

$$\frac{n(n+1)}{4}$$

We also need for this to give whole number solutions for it to make sense in the context of our problem.

In order for $\frac{n(n+1)}{4}$ to give whole number solutions, *n* must be of the form 4*k* or 4*k*+3 for some nonnegative integer *k*.



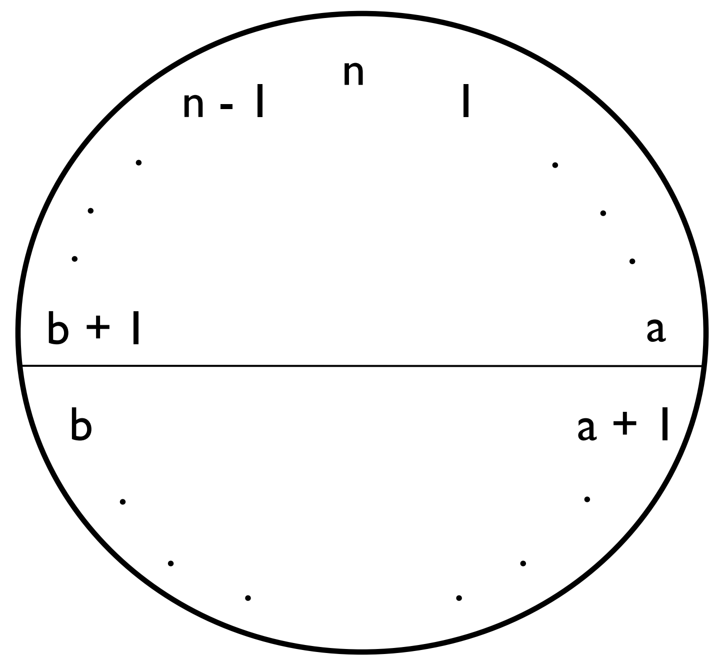
For a cut from a to b to be a solution, the following must be true:

$$\sum_{i=1}^{b} i - \sum_{i=1}^{a} i = \frac{1}{2} \cdot \frac{n(n+1)}{2}$$

$$\frac{b(b+1)}{2} - \frac{a(a+1)}{2} = \frac{n(n+1)}{4}$$

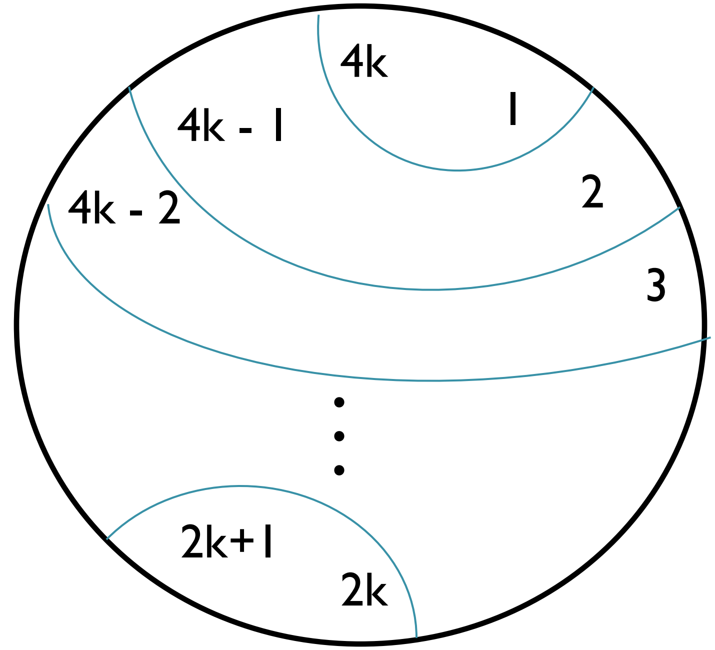
$$b^2 + b - a^2 - a = \frac{n(n+1)}{2}$$

$$(b-a)(b+a+1) = \frac{n(n+1)}{2}$$

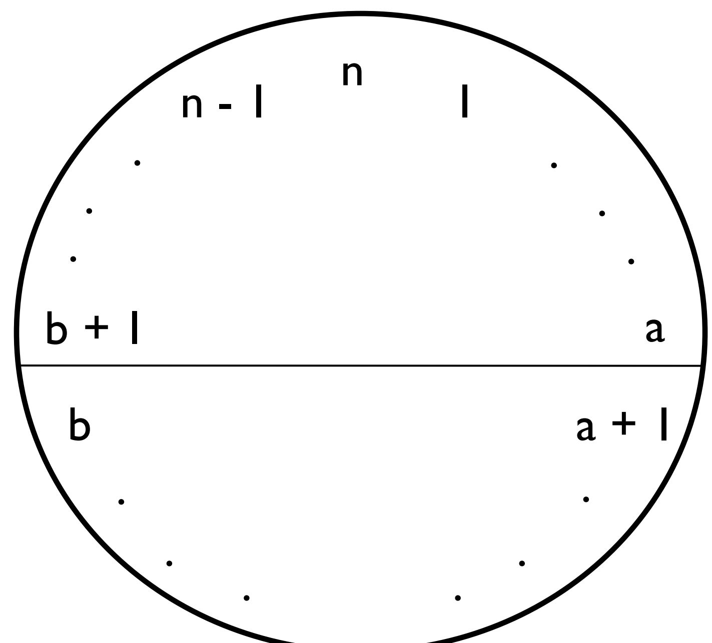


Let’s now consider the case for when n = 4k

Each number on the clock can be paired with another number so that a total of 2k pairs is obtained, with each pair containing a sum of 4k+1. So, to find a sequence whose sum is half of the total sum of 4k, we simply take the first half of these pairs, in other words, the first k of these pairs.




Selecting the first k of these pairs corresponds to solutions of a = k and b = 3k.




Checking these solutions in our equation we see:

$$(b-a)(b+a+1)=\frac{n(n+1)}{2}$$

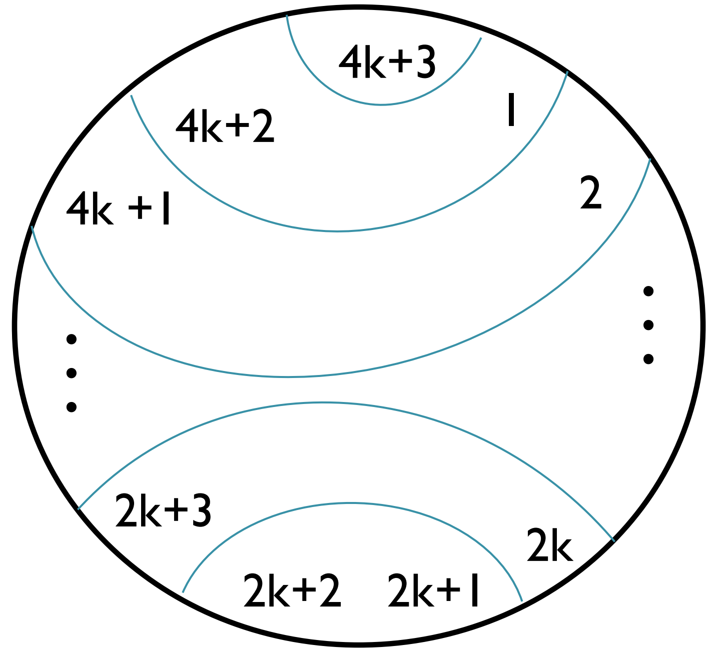
$$(3k-k)(3k+k+1)=\frac{4k(4k+1)}{2}$$

$$(2k)(4k+1)=(2k)(4k+1)$$

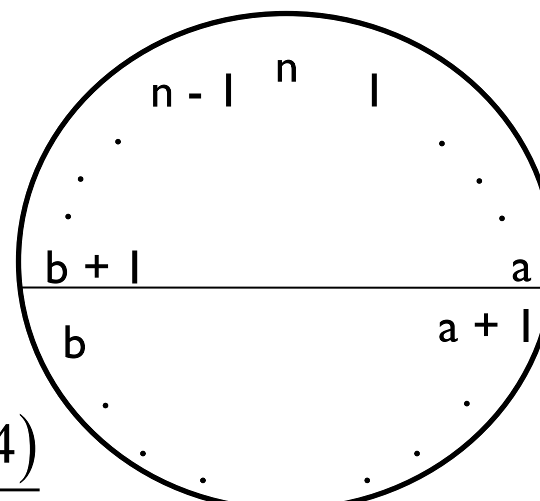
n = 4k+3 Case

This case is similar to the previous case in that it also may be solved by pairing.

We end up with 2k+2 sets each containing a sum of 4k+3. To find a sequence whose sum is one half of the total sum, we take the first half, or the first k+1, of these pairs. The first k+1 sets consist of the integers from 1 to k with their respective pairs, and 4k+3 by itself, meaning a=k. a=k corresponds to b=3k+2 since a and b+1 are a pair adding up to 4k+3.




Checking these solutions in our equation we see:




$$(b-a)(b+a+1)=\frac{n(n+1)}{2}$$

$$(3k+2-k)(3k+2+k+1)=\frac{(4k+3)(4k+4)}{2}$$

$$(2k+2)(4k+3)=(2k+2)(4k+3)$$

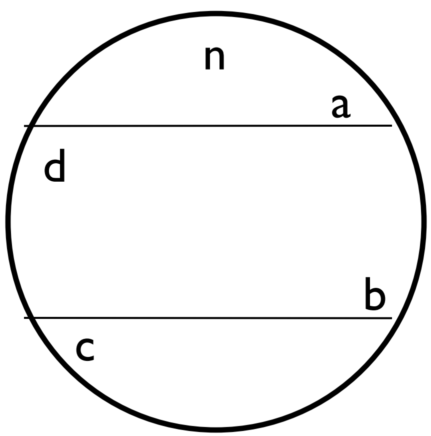
# Conclusion for the One Cut Case

- Natural numbers of the form 4k and 4k+3 will always have at least one solution and natural numbers of the form 4k+1 and 4k+2 will never have solutions (for nonnegative integers k).

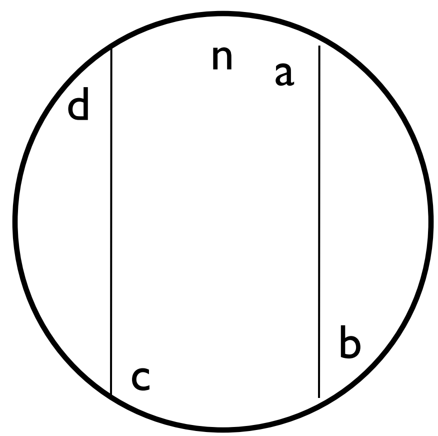
# The Two Cut Case

- For a case where our n-faced clock is divided with two cuts, we have three separate cases:

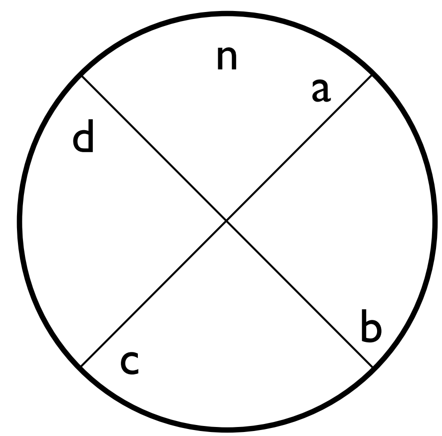
Case A




Case B




Case C



# Case A

For Case A to have solutions we need:

$$\sum_{i=1}^{c} i - \sum_{i=1}^{b} i = \frac{1}{3}\sum_{i=1}^{n} i$$

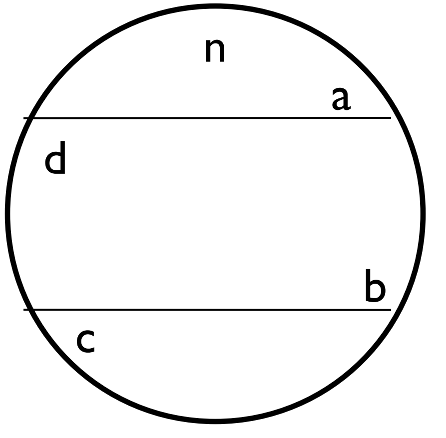
$$(c-b)(c+b+1) = \frac{n(n+1)}{3}$$

and:

$$\sum_{i=1}^{d} i - \sum_{i=1}^{a} i = \frac{2}{3}\sum_{i=1}^{n} i$$

$$(d-a)(d+a+1) = \frac{2n(n+1)}{3}$$

Case A



# Case B

For Case B to have solutions we need:

$$\sum_{i=1}^{b} i - \sum_{i=1}^{a} i = \frac{1}{3}\sum_{i=1}^{n} i$$

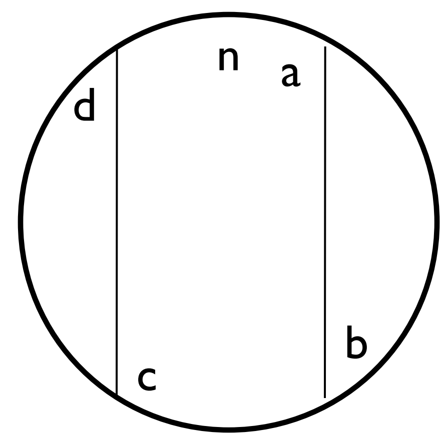
$$(b-a)(b+a+1) = \frac{n(n+1)}{3}$$

and:

$$\sum_{i=1}^{d} i - \sum_{i=1}^{c} i = \frac{1}{3}\sum_{i=1}^{n} i$$

$$(d-c)(d+c+1) = \frac{n(n+1)}{3}$$

Case B



# Case C

For Case C to have solutions we need:

$$\sum_{i=a+1}^{b} i = \sum_{i=b+1}^{c} i = \sum_{i=c+1}^{d} i = \frac{1}{4}\sum_{i=1}^{n} i$$

$$\sum_{i=a+1}^{b} i = \frac{1}{4}\sum_{i=1}^{n} i$$

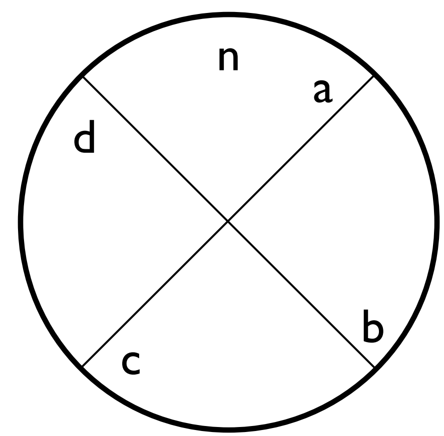
$$b^2 + b - \left(a^2 + a\right) = \frac{n\left(n+1\right)}{4}$$

$$b^2 + b + \frac{1}{4} - \left(a^2 + a + \frac{1}{4}\right) = \frac{n\left(n+1\right)}{4}$$

$$4b^2 + 4b + 1 - \left(4a^2 + 4a + 1\right) = n\left(n+1\right)$$

$$\left(2b+1\right)^2 - \left(2a+1\right)^2 = n\left(n+1\right)$$

Case C



# Case C

The equation from the previous slide corresponds to two additional equations for b, c, and d:

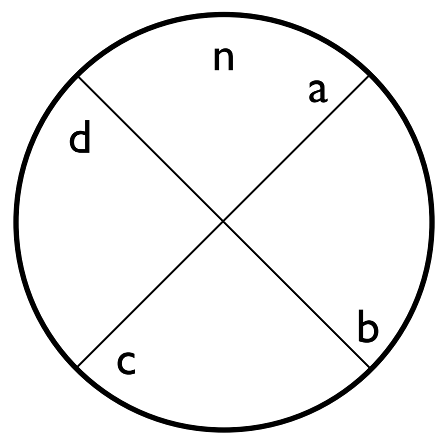
$$(2b+1)^2-(2a+1)^2=n(n+1)$$

$$(2c+1)^2-(2b+1)^2=n(n+1)$$

$$(2d+1)^2-(2c+1)^2=n(n+1)$$

These three equations are four squares in arithmetic progression. By Fermat's four squares theorem, no integer solutions will be found for these equations making it impossible for solutions to our problem to occur under Case C.

Case C

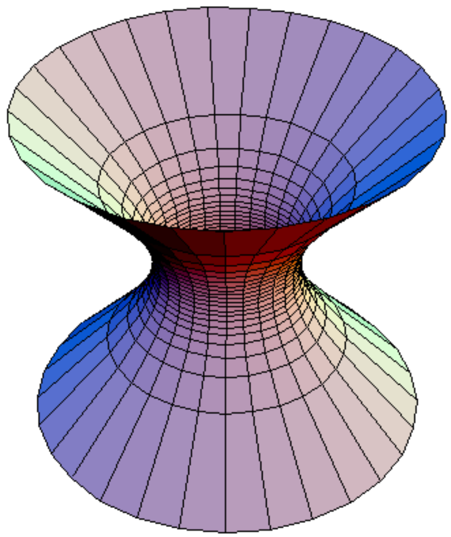


# Finding Solution Families

- Examining one of equations from Case A, we see that it is the equation of a hyperboloid of one sheet.

$$(c-b)(c+b+1)=\frac{n(n+1)}{3}$$

$$c^2-b^2+c-b=\frac{n(n+1)}{3}$$



# Finding Solution Families

- One property that we know hyperboloids of one sheet have is that every point on the hyperboloid has two lines passing through said point that lie completely on the hyperboloid.

# Finding Solution Families

- We are able to find a solution family for any solution we have by setting up parametric equations.
- For example, in Click and Clack's original problem, a=2, b=4, c=8, and d=10 were a solution when n=12.

# Finding Solution Families

- Now we set: $a = 2 + \alpha t$

$$b = 4 + \beta t$$

$$c = 8 + \gamma t$$

$$d = 10 + \delta t$$

$$n = 12 + \varepsilon t$$

# Finding Solution Families

- By using our equations from Case A, we see:

$$0=(c-b)(c+b+1)-\frac{n(n+1)}{3}$$

$$0=\left(-\beta^2+\gamma^2-\frac{\varepsilon^2}{3}\right)t^2+\left(-9\beta+17\gamma-\frac{25\varepsilon}{3}\right)t$$

and

$$0=(d-a)(d+a+1)-\frac{2n(n+1)}{3}$$

$$0=\left(-\alpha^2+\delta^2-\frac{2\varepsilon^2}{3}\right)t^2+\left(-5\alpha+21\delta-\frac{50\varepsilon}{3}\right)t$$

# Finding Solution Families

$$0 = \left(-\alpha^2 + \delta^2 - \frac{2\varepsilon^2}{3}\right)t^2 + \left(-5\alpha + 21\delta - \frac{50\varepsilon}{3}\right)t$$

$$0 = \left(-\beta^2 + \gamma^2 - \frac{\varepsilon^2}{3}\right)t^2 + \left(-9\beta + 17\gamma - \frac{25\varepsilon}{3}\right)t$$

- By setting these coefficients equal to zero and solving in terms of epsilon, we see that:

| | **1** | **2** |
|---|---|---|
| $\alpha$ | $\varepsilon/6$ | $73\varepsilon/312$ |
| $\beta$ | $\varepsilon/3$ | $121\varepsilon/312$ |
| $\gamma$ | $2\varepsilon/3$ | $217\varepsilon/312$ |
| $\delta$ | $5\varepsilon/6$ | $265\varepsilon/312$ |

# Finding Solution Families

- From the previous table, by multiplying each row by the least common multiple of the denominators and substituting into our original parametric equations, we get four solution families:

| | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|---|---|
| 1 | t+2 | 2t+4 | 4t+8 | 5t+10 | 6t+12 |
| 2 | 52t+2 | 104t+4 | 217t+8 | 265t+10 | 312t+12 |
| 3 | 73t+2 | 121t+4 | 208t+8 | 260t+10 | 312t+12 |
| 4 | 73t+2 | 121t+4 | 217t+8 | 265t+10 | 312t+12 |

# Finding Solution Families

- From the previous table, by multiplying each row by the least common multiple of the denominators and substituting into our original parametric equations, we get four solution families:

| | $\alpha$ | $\beta$ | $\gamma$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|---|---|
| 1 | t | 2t | 4t | 5t | 6t |
| 2 | 52t+2 | 104t+4 | 217t+8 | 265t+10 | 312t+12 |
| 3 | 73t+2 | 121t+4 | 208t+8 | 260t+10 | 312t+12 |
| 4 | 73t+2 | 121t+4 | 217t+8 | 265t+10 | 312t+12 |

# Other Solution Families for the Two Cut Case

| Case A | Case B |
| --- | --- |
| 6k | 15k+5 |
| 6k+5 | 15k+9 |
| 36k+9 | 36k+9 |
| 36k+26 | 36k+26 |
| 72k+27 | 168k+20 |
| 72k+44 | 168k+147 |
| 90k+9 | 180k+170 |
| 90k+80 | 231k+98 |
| 120k+44 | 288k+207 |
| 120k+75 | 420k+329 |
| 82386k+351 | 624k+584 |

# Caleb's Conclusion to Click and Clack's Clock

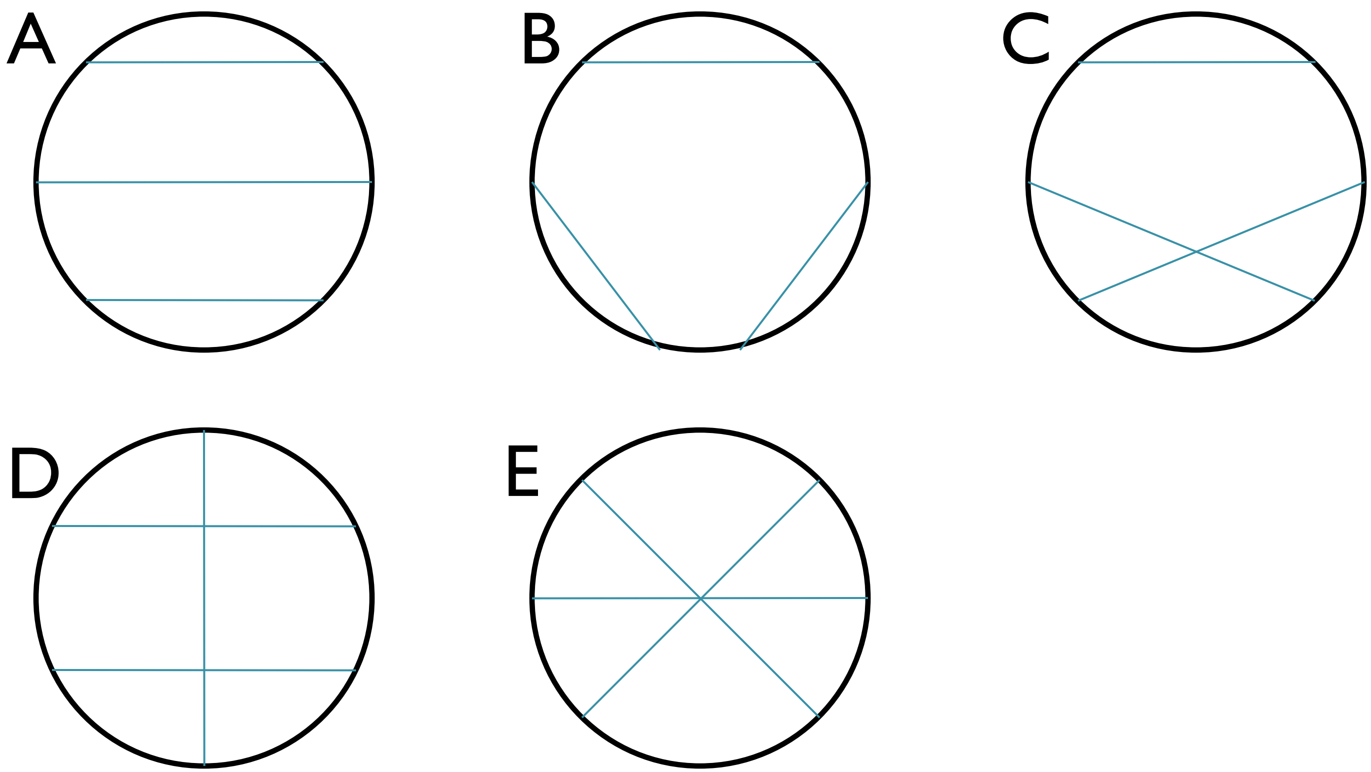
- Through our methods for finding solutions, we have created families of solutions covering roughly 1/2 of the integers, ruled out 1/3 of the integers, leaving roughly 16% remaining.
- We are in the process of (hopefully) showing that there are no families of non-solutions other than $n=3k+1$.
- We think similar methods will show that the limit of the number of solutions as n approaches infinity will be 2/3 of the integers.

# Increasing the Number of Cuts

- We have conclusions to both the one-cut and two-cut cases of our problem, but what can we say when the number of cuts is increased?

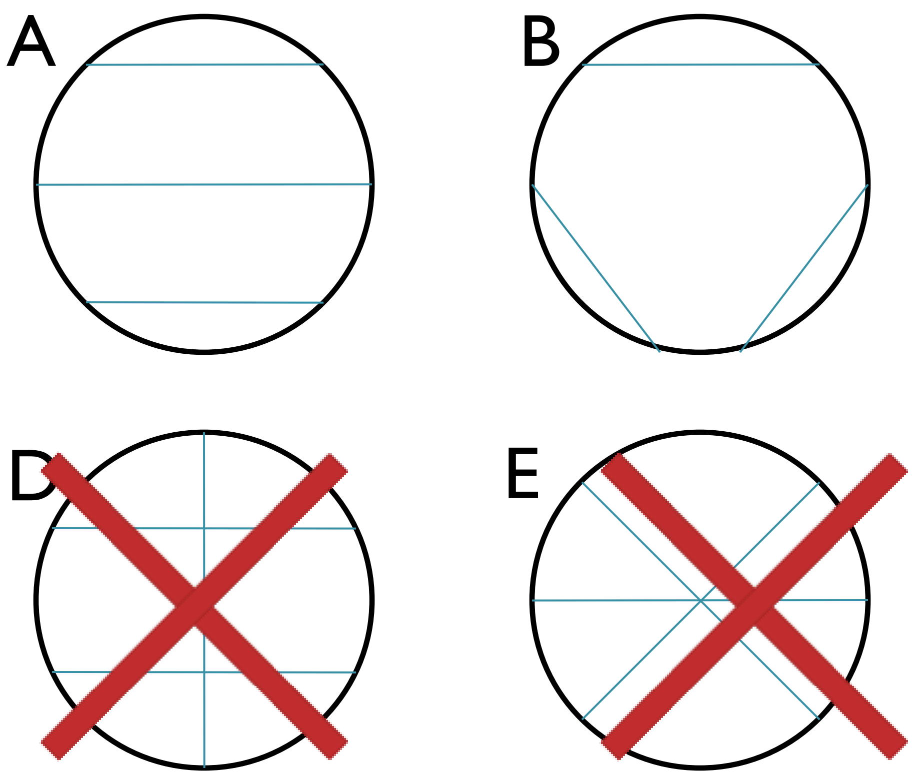
# Making More Than 2 Cuts

- There are 5 unique ways to make 3 cuts across a clock:



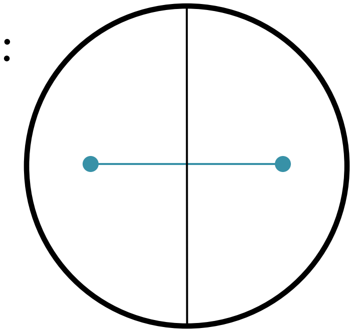
# Making More Than 2 Cuts

- Three of these 5 ways produce four squares in arithmetic progression, making them impossible for our problem.

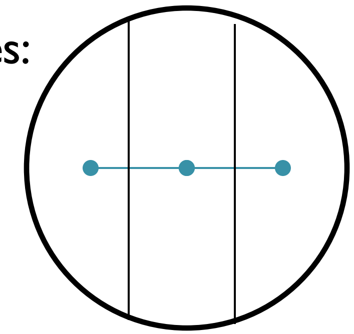


For t cuts, the number of ways to slice the clock without having four squares in arithmetic progression follows the pattern for the number of unlabeled planar trees with t+1 nodes.

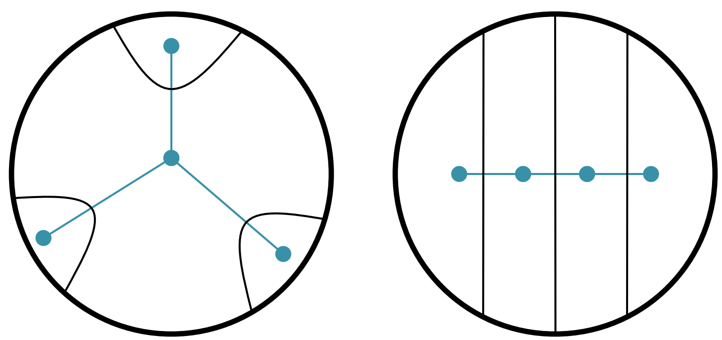
Two nodes:



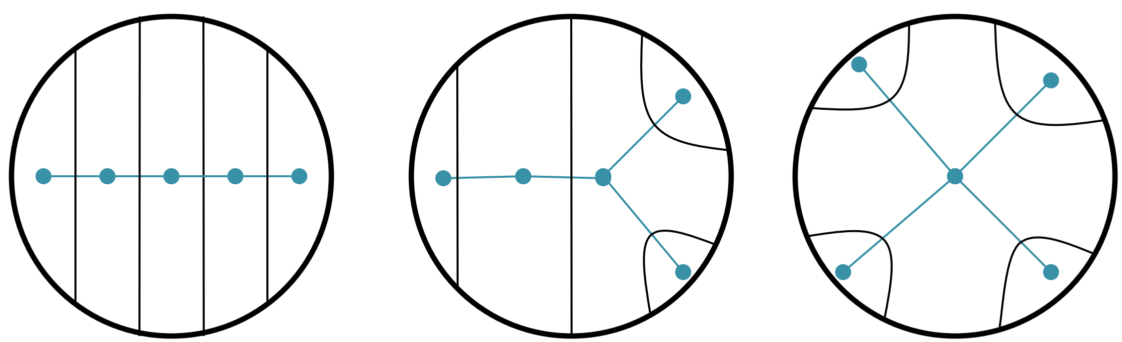
Three nodes:



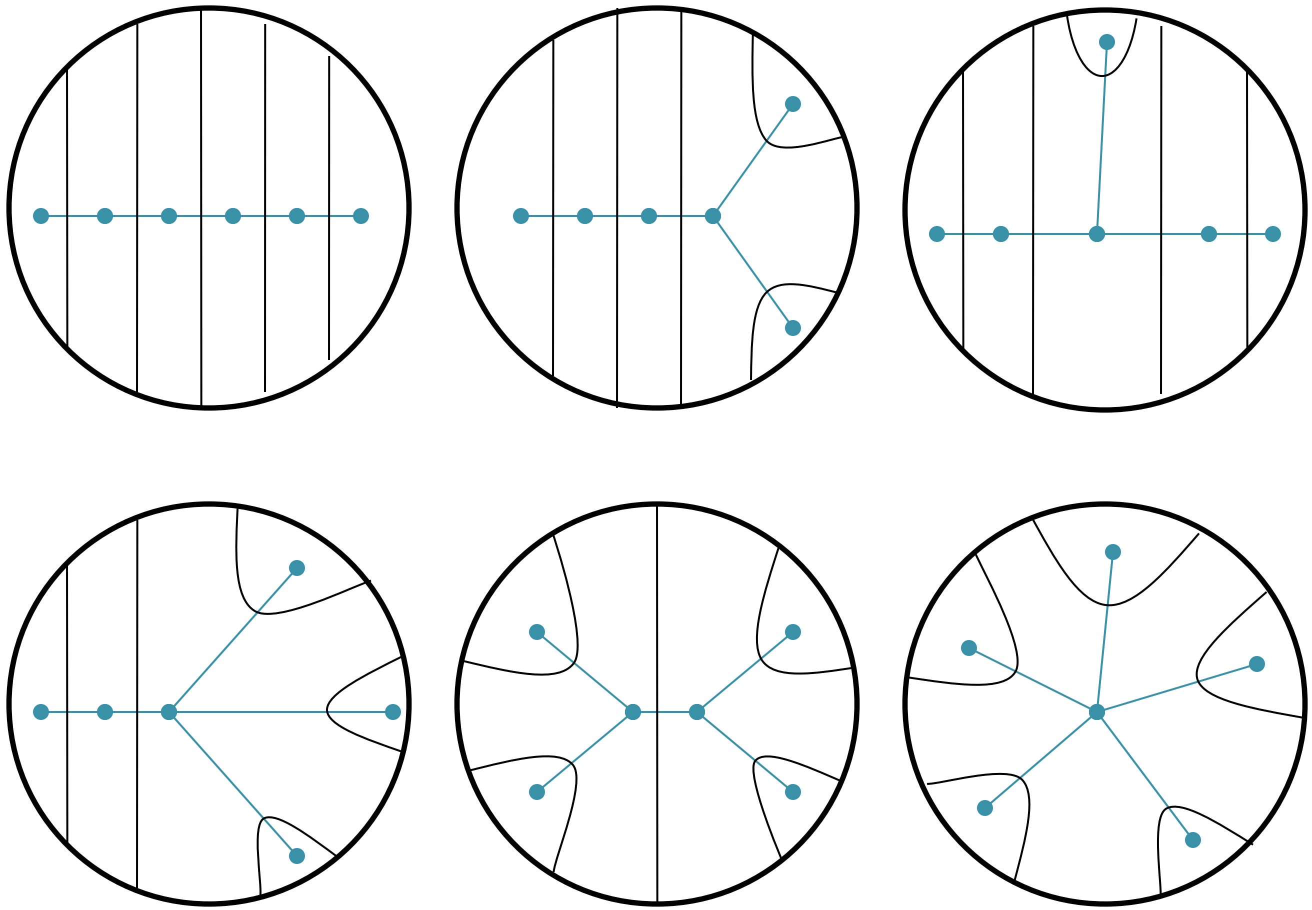
Four nodes:



Five nodes:



Six nodes:



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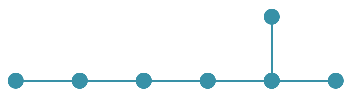

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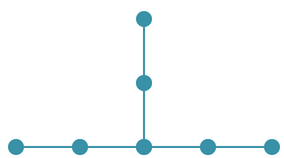

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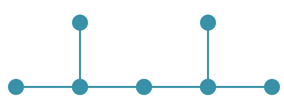

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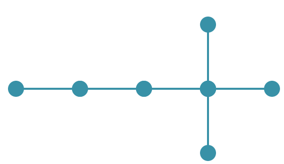

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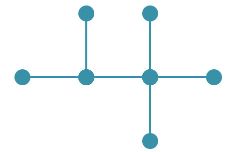

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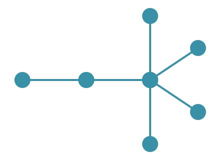

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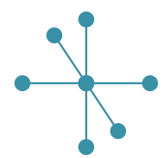

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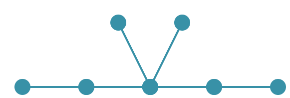

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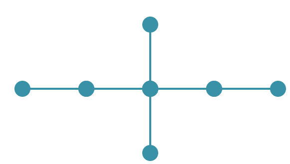

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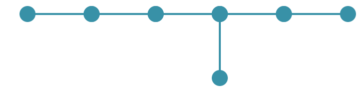

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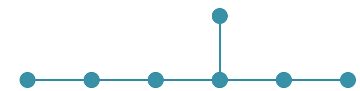

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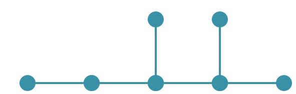

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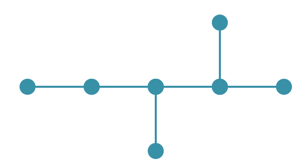

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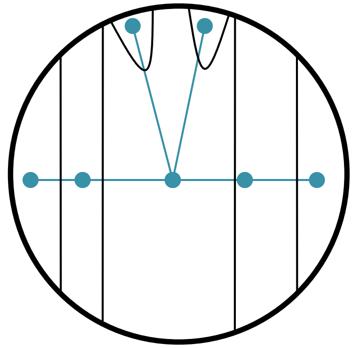

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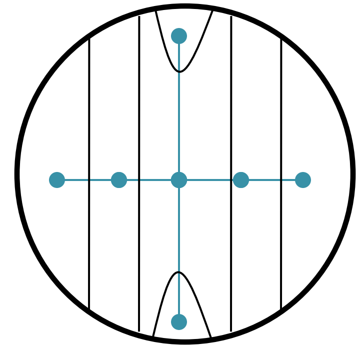

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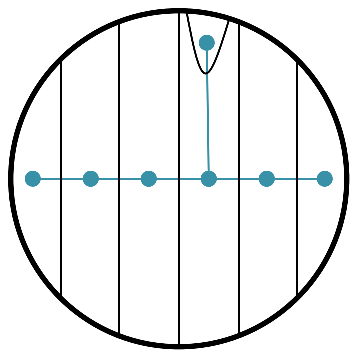

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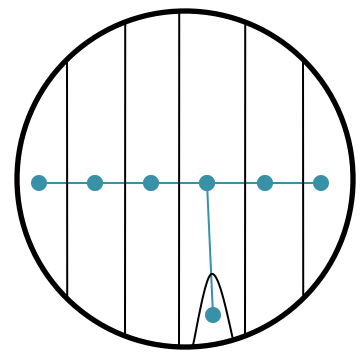

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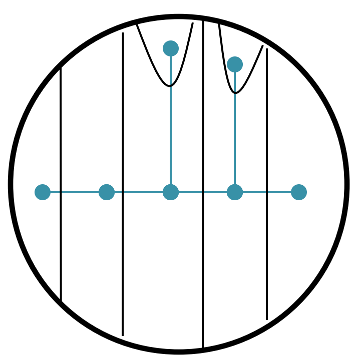

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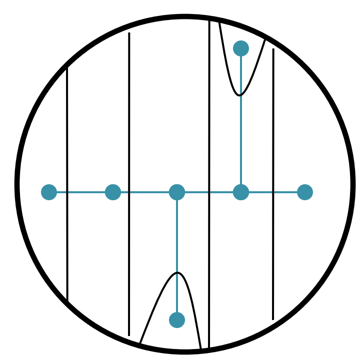
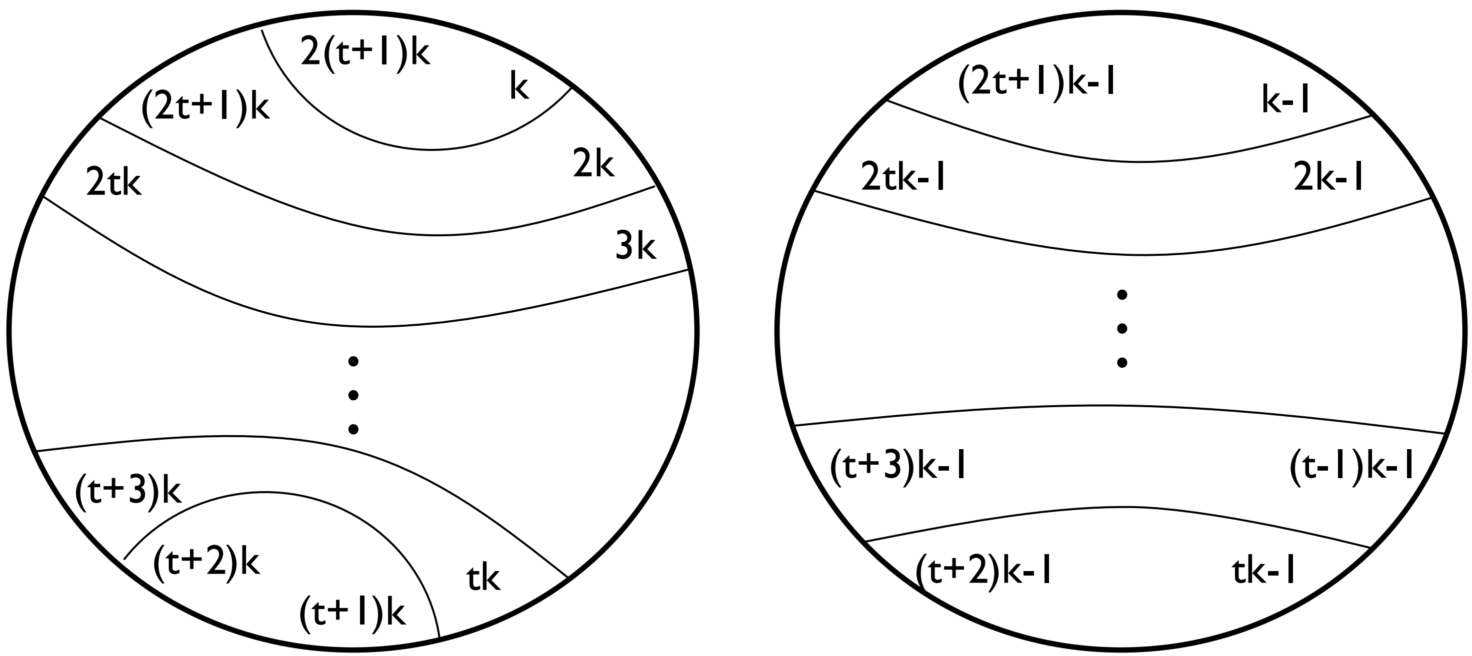


# Theorem

- When t cuts are made across an n-faced clock, n of the form $n=2(t+1)k$ and $n=2(t+1)k-1$, for some positive integer k, will always have solutions.

# Sketch of Proof for Theorem

- By pairing, we can show that when (t+1)k pieces are formed, the numerals around the clock may be grouped in such a way that they add to the correct summation.



# Theorem

- Any time $p^k - 1$ cuts are made across an *n*-faced clock, where p is some prime, existence of solutions can be determined for any positive integer *n*.

# Sketch of Proof for Theorem

- If $p^k - 1$ non-overlapping slices are made into the clock, $p^k$ pieces are formed.
- Only n of the form $p^k s$ or $p^k s - 1$ are eligible to have solutions because other n do not produce summations divisible by $p^k$.

# Sketch of Proof for Theorem

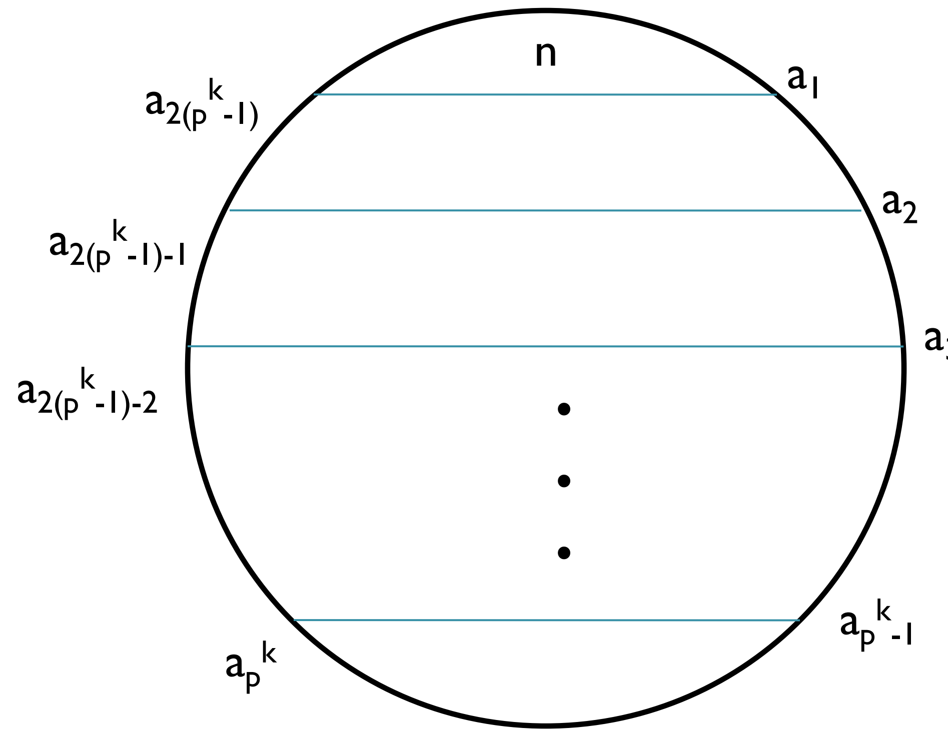
- For an n to be eligible, we need $\frac{n(n+1)}{2p^k}$ to give a whole number solution.
- In other words, we need

$$n(n+1) \equiv 0 \bmod 2p^k$$

- The only place this will occur is at $n=p^k$ and $n=p^k-1$.

# Proof Sketch Cont.

Once again, we use the idea of pairing numbers together. By doing it the way shown at the right, we can show that there is always at least one solution for cases where $n=p^k s$ or $n=p^k s+p^k-1$



# In Summary

- Solution families have been found that encompass just over half of the integers for Click and Clack's Clock problem. Another one third have been ruled out as solutions because of their summations.
- We are in the process of proving that there are no families of non-solutions.
- We hope to be able to show that the limit of the number of solutions approaches two thirds.

# In Summary

- Existence of solutions may be determined for any integer n when $p^k-1$ cuts are made across the clock.
- When t cuts are made across an n-faced clock, $n=2(t+1)k$ and $n=2(t+1)k-1$ will always have solutions.
- The number of ways to divide a clock using t non-overlapping slices follows the pattern for the number of unlabeled planar trees with t+1 nodes.

# Future Ventures

- Investigate non-existence of solutions to crossing cuts cases.
- Investigate our conjecture that the limit of increasing cuts can be determined.

$$\frac{n(n+1)}{2l}$$