Understanding when Finite Local Rings share Zero Divisor Graphs

Sarah Fletcher

MSU REU

30 July 2008

Zero Divisor Graph

Let R be a ring and let $Z^*(R)$ denote the set of zero divisors of R.

Definition

$$\Gamma(R) = (Z^*(R), \{(a, b)|a, b \in Z^*(R), ab = 0\})$$

Zero Divisor Graph

Let R be a ring and let $Z^*(R)$ denote the set of zero divisors of R.

Definition

$$\Gamma(R) = (Z^*(R), \{(a,b) | a, b \in Z^*(R), ab = 0\})$$

Example:
$R = \mathbb{Z}_{12}$

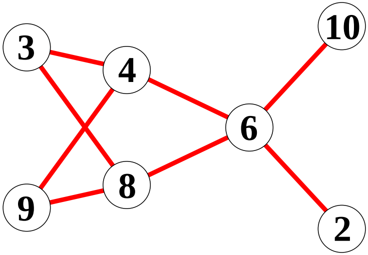
Zero Divisor Graph

Let R be a ring and let $Z^*(R)$ denote the set of zero divisors of R.

Definition

$$\Gamma(R) = (Z^*(R), \{(a,b) | a, b \in Z^*(R), ab = 0\})$$

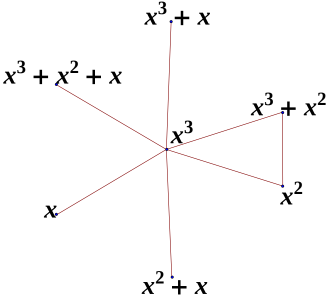
Example:
$R = \mathbb{Z}_{12}$



Local Rings of Order 16, $|\mathcal{M}| = 8$

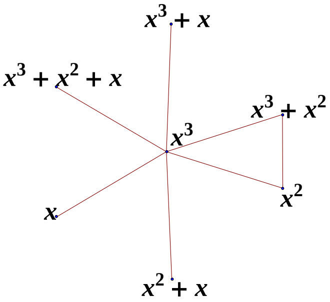
- Characteristic 2: $\mathbb{Z}_2[x]/(x^4)$, $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$, $\mathbb{Z}_2[x,y]/(x^3, xy, x^2 - y^2)$, $\mathbb{Z}_2[x,y]/(x^2, y^2)$,and $\mathbb{Z}_2[x,y,z]/(x,y,z)^2$
- Characteristic 4: $\mathbb{Z}_4[x]/(x^2, 2x)$, $\mathbb{Z}_4[x]/(x^2)$, $\mathbb{Z}_4[x]/(x^2 - 2x)$, $\mathbb{Z}_4[x,y]/(x^3, y^2, xy, x^2 - 2)$, $\mathbb{Z}_4[x,y]/(x^3, x^2 - 2, x^2 - y^2, xy)$,and $\mathbb{Z}[x,y]/(x^2, y^2, xy - 2)$
- Characteristic 8: $\mathbb{Z}_8[x]/(x^3, x^2 - 4, 2x)$, and $\mathbb{Z}_8[x]/(x^3, x^2 - 4, 2x)$
- Characteristic 16: $\mathbb{Z}_{16}$

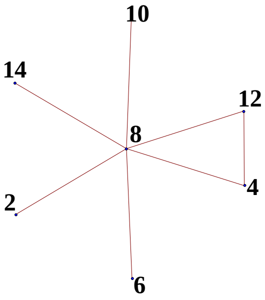
$\mathbb{Z}_2[x]/(x^4)$

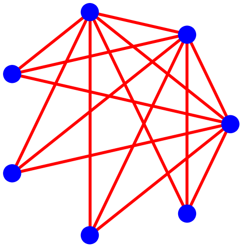


$\mathbb{Z}_2[x]/(x^4)$

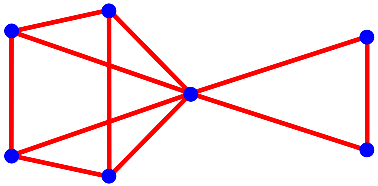
$\mathbb{Z}_{16}$



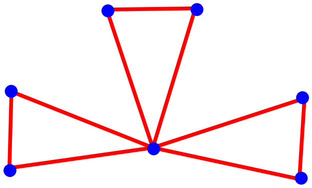




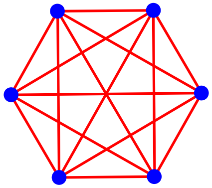
- $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$
- $\mathbb{Z}_4[x,y]/(x^3, x^2-2, xy, y^2)$
- $\mathbb{Z}_4[x]/(x^3, 2x)$
- $\mathbb{Z}_8[x]/(x^2, 2x)$



- $\mathbb{Z}_2[x,y]/(x^3, xy, x^2 - y^2)$
- $\mathbb{Z}_4[x,y]/(x^3, xy, x^2 - 2, x^2 - y^2)$
- $\mathbb{Z}_4[x]/(x^2 - 2x)$
- $\mathbb{Z}_8[x]/(x^3, x^2 - 4, 2x)$

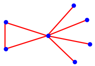
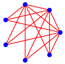
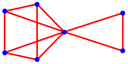
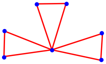
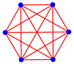


- $\mathbb{Z}_2[x, y]x^2, y^2)$
- $\mathbb{Z}_4[x, y]/(x^2, y^2, xy - 2)$
- $\mathbb{Z}_4[x]/(x^2)$



- $\mathbb{Z}_2[x, y, z]/(x, y, z)^2$
- $\mathbb{Z}_{16}$

Local Rings of Order 16, $|\mathcal{M}| = 8$

	Characteristic 2
	$\mathbb{Z}_2[x]/(x^4)$
	$\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$
	$\mathbb{Z}_2[x,y]/(x^3, xy, x^2 - y^2)$
	$\mathbb{Z}_2[x,y]/(x^2, y^2)$
	$\mathbb{Z}_2[x,y,z]/(x,y,z)^2$

Galois Rings

Let τ^* be the unique cyclic multiplicative subgroup of order $p^r - 1$ in the Galois ring

$$G(p^k, r) = \mathbb{Z}_{p^k}[x]/(f).$$

Galois Rings

Let τ^* be the unique cyclic multiplicative subgroup of order $p^r - 1$ in the Galois ring

$$G(p^k, r) = \mathbb{Z}_{p^k}[x]/(f).$$

The **Teichmüller set** is $\tau = \tau^* \cup \{0\}$.
Then any element $a \in G(p^k, r)$ can be uniquely represented as

$$a = a_0 + a_1 p + a_2 p^2 + \cdots + a_{k-1} p^{k-1}.$$

Define $\varphi : G(p^k, r) \to \mathbb{F}_{p^r}[y]/(y^k)$ so that

$$a \mapsto \pi(a_0) + \pi(a_1)y + \pi(a_2)y^2 + \cdots + \pi(a_{k-1})y^{k-1}$$

where the a_i come from the Teichmüller representation of a.

Define $\varphi : G(p^k, r) \to \mathbb{F}_{p^r}[y]/(y^k)$ so that

$$a \mapsto \pi(a_0) + \pi(a_1)y + \pi(a_2)y^2 + \cdots + \pi(a_{k-1})y^{k-1}$$

where the a_i come from the Teichmüller representation of a.

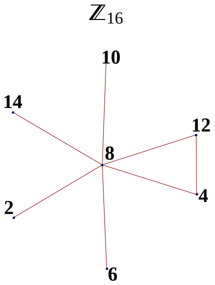
Theorem

φ induces a graph isomorphism between $\Gamma(G(p^k, r))$ and $\Gamma(\mathbb{F}_{p^r}[y]/(y^k))$.

$$\varphi(a) = \pi(a_0) + \pi(a_1)y + \pi(a_2)y^2 + \cdots + \pi(a_{k-1})y^{k-1}$$

$$\varphi(a) = \pi(a_0) + \pi(a_1)y + \pi(a_2)y^2 + \cdots + \pi(a_{k-1})y^{k-1}$$

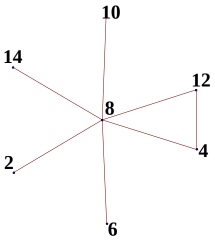
Example: $G(2^4, 1) = \mathbb{Z}_{16}$
Note that $\tau = \{0, 1\}$.



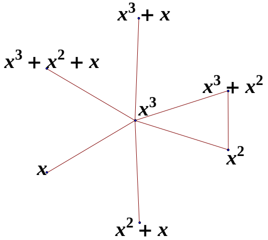
$\varphi(a) = \pi(a_0) + \pi(a_1)y + \pi(a_2)y^2 + \cdots + \pi(a_{k-1})y^{k-1}$

Example: $G(2^4, 1) = \mathbb{Z}_{16}$
Note that $\tau = \{0, 1\}$.

$\mathbb{Z}_{16}$



$\mathbb{Z}_2[x]/(x^4)$



Finite Local Rings

Let $G(p^k, r)[x_1, \ldots, x_n]/I$ be a finite local ring and let

$$R = G(p^k, r)[x_1, \ldots, x_n]$$

Finite Local Rings

Let $G(p^k, r)[x_1, \ldots, x_n]/I$ be a finite local ring and let

$$R = G(p^k, r)[x_1, \ldots, x_n]$$

$$S = F_{p^r}[y]/y^k[x_1, \ldots, x_n].$$

Finite Local Rings

Let $G(p^k, r)[x_1, \dots, x_n]/I$ be a finite local ring and let

$$R = G(p^k, r)[x_1, \dots, x_n]$$

$$S = F_{p^r}[y]/y^k[x_1, \dots, x_n].$$

Define $\psi : R \to S$ by

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mapsto \varphi(a_\alpha) x^\alpha.$$

Finite Local Rings

Let $G(p^k, r)[x_1, \ldots, x_n]/I$ be a finite local ring and let

$$R = G(p^k, r)[x_1, \ldots, x_n]$$

$$S = F_{p^r}[y]/y^k[x_1, \ldots, x_n].$$

Define $\psi : R \to S$ by

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mapsto \varphi(a_\alpha) x^\alpha.$$

Making Things Finite Again

Given the finite local ring $G(p^k, r)[x_1, \ldots, x_n]/I$, let

$$Q = (\{x_i^m | x_i^m \in I\})$$

Making Things Finite Again

Given the finite local ring $G(p^k, r)[x_1, \ldots, x_n]/I$, let

$$Q = (\{x_i^m | x_i^m \in I\})$$

$$R_Q = G(p^k, r)[x_1, \ldots, x_n]/Q \qquad S_Q = F_{p^r}[y]/y^k[x_1, \ldots, x_n]/Q.$$

Making Things Finite Again

Given the finite local ring $G(p^k, r)[x_1, \ldots, x_n]/I$, let

$$Q = (\{x_i^m | x_i^m \in I\})$$

$$R_Q = G(p^k, r)[x_1, \ldots, x_n]/Q \qquad S_Q = F_{p^r}[y]/y^k[x_1, \ldots, x_n]/Q.$$

Define $\psi_Q : R_Q \to S_Q$ by

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mapsto \varphi(a_\alpha) x^\alpha.$$

Making Things Finite Again

Given the finite local ring $G(p^k, r)[x_1, \ldots, x_n]/I$, let

$$Q = (\{x_i^m | x_i^m \in I\})$$

$$R_Q = G(p^k, r)[x_1, \ldots, x_n]/Q \qquad S_Q = F_{p^r}[y]/y^k[x_1, \ldots, x_n]/Q.$$

Define $\psi_Q : R_Q \to S_Q$ by

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mapsto \varphi(a_\alpha) x^\alpha.$$

Note that we can find an I' so that

$$R_Q/I' \cong R/I.$$

An Idea

$$J = \{\psi_Q(a)\psi_Q(b) | ab \in I'\} \subseteq S_Q$$

An Idea

$$J = \{\psi_Q(a)\psi_Q(b) | ab \in I'\} \subseteq S_Q$$

Compare:

$$\Gamma(R_Q/I') \text{ and } \Gamma(S_Q/(J)).$$