The Annihilator Graph of a Ring

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Background

In 2002, Mulay introduced the idea of an analog to the zero divisor graph of a finite ring for an infinite ring. This graph would have annihilators of elements for nodes and edges if the generators of the annihilators were zero divisors. This opened up infinite rings to the same type of research going on for finite rings. We examine some of those topics here.

Evolution

The motivation for this research is to look at old problems from new angles and possibly discover new theorems. The zero divisor graph is a graph theoretical tool we can use to study finite rings, but they don't work well for infinite rings. The problem is that the zero divisor graphs for infinite rings become very dense and complicated extremely quickly.

Evolution

To combat this problem, we define an equivalence relation on the infinite ring and make the nodes of our new graph representatives from each class. This didn't solve all of our problems though. We still ended up with infinite graphs, but certain structural consistencies arose from these infinite graphs.

Evolution

These consistencies lead us to the final incarnation of the annihilator graph of a ring. We will explore them, and their properties here.

Preliminary Terms and Definitions

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We define a partial order on M_I: if s, t $\in M_I$, $s = \prod_{i=1}^{n} x_i^{b_i}$ and $t = \prod_{i=1}^{n} x_i^{a_i}$ we say $s \leq t$ if and only if there exists j,k such that $a_j < b_j$ and $a_k > b_k$.

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Example: The cartesian plane.

Preliminary Terms and Definitions

Definition: Let A and B be subsets of M_l. The *monomial set product* of A and B is another subset of M_l given by

$$AB = \{ab \mid a \in A \text{ and } b \in B\}$$

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We will refer to the monomial set annihilator as the annihilator unless otherwise indicated. Furthermore, the annihilator of the set containing a single element, Ann({a}) will be written as Ann(a).

Example

Let $R = D[X,Y]/(X^3,Y^4)$. Let $A=\{x^2y, xy^3\}$, $B=\{y^2, xy, x^2\}$, and $B'=\{x^2y^3\}$.

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Then $AB=\{0, x^2y^3, x^3y^2\}$ and $AB'=\{0\}$.

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The Annihilator Graph

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Definition: In a partially ordered set, a poset, any two elements which are not partially ordered, are said to be *incomparable.*

Definition: An *antichain* is a set of elements under a partial order such that any two elements are incomparable.

The Annihilator Graph

Let I be a monomial ideal, and let $R=D[x_1,\ldots,x_n]/I$. The *annihilator graph* of R, $\Gamma_a(R)$, is the graph whose vertices are the nontrivial equivalence classes, [A], of ~ defined on M_I, and whose edges are the ordered pairs ([A],[B]), of nontrivial equivalence classes satisfying AB=0.

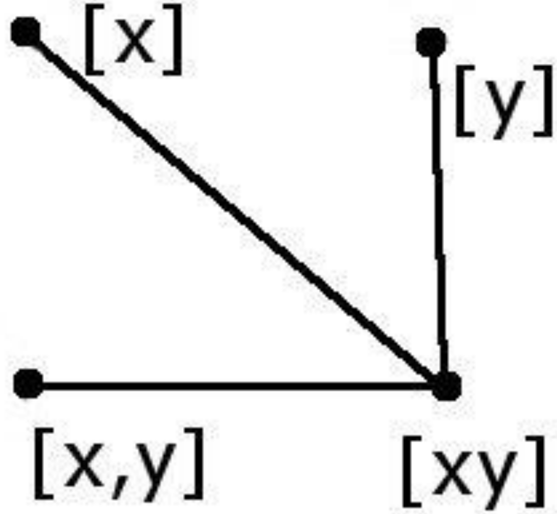
Examples

Consider the ring $R = D[x,y]/(x^2,y^2)$. We see the only antichains with unique annihilators are the elements x, y, xy, and the set {x,y}. Thus, $\Gamma_a(R)$ has 4 nodes, and looks like

Examples

$$R = D[x,y] / (x^2, y^2)$$

$$\Gamma_a(R) =$$

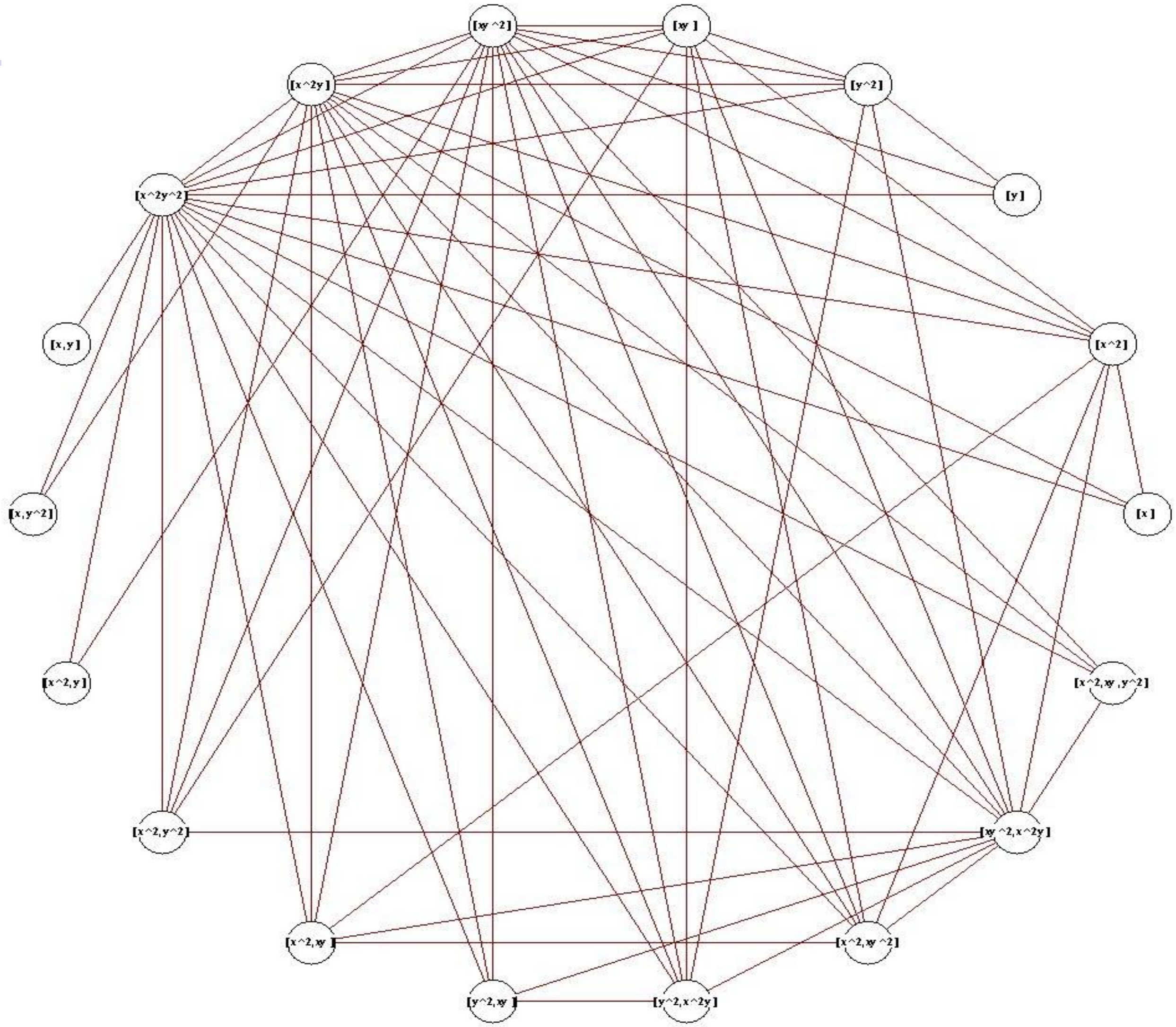


Examples

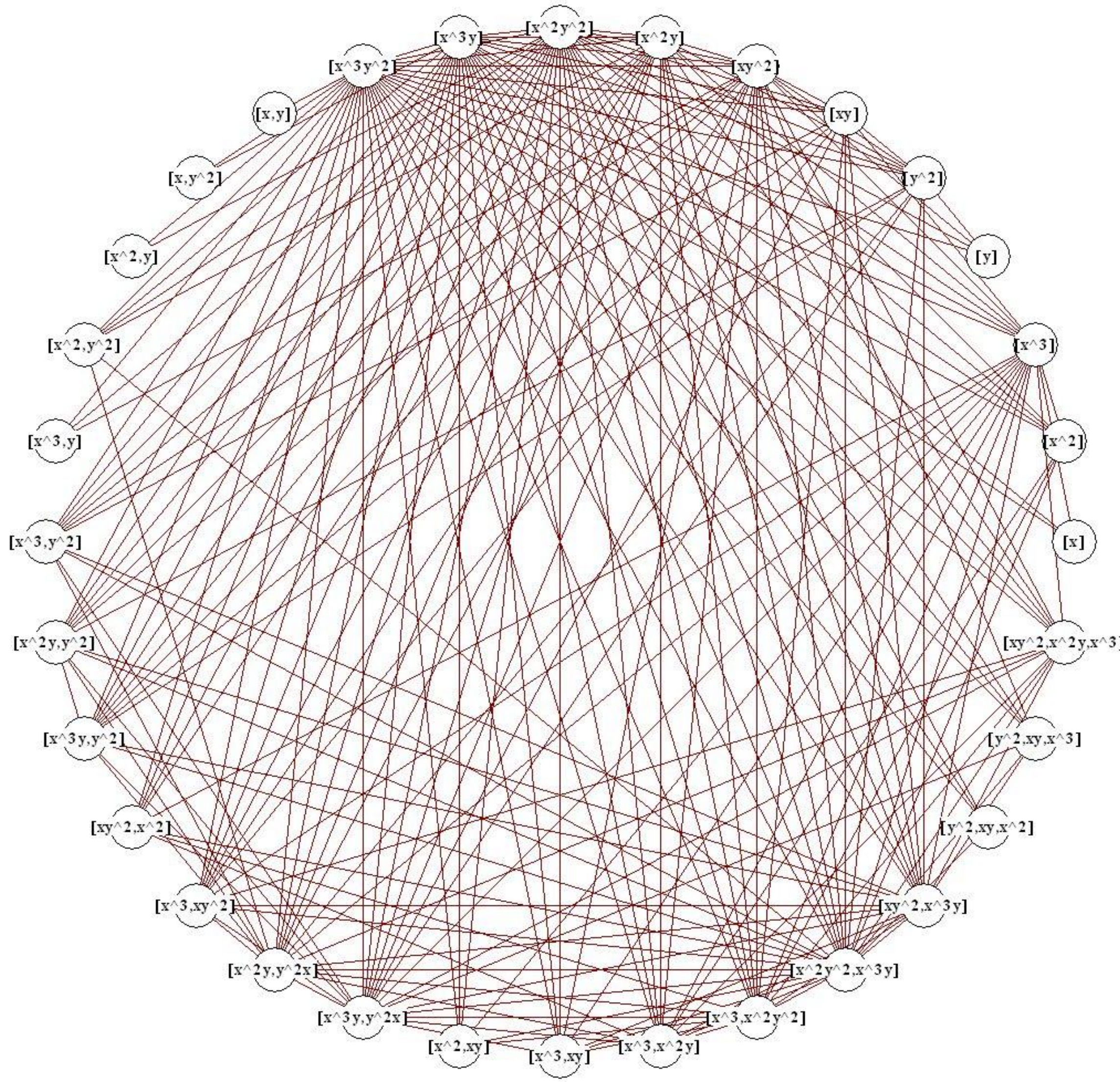
The following two examples are for illustrative purposes only, since they are too complicated to derive any useful information directly from. They are for the rings $D[x,y]/(x^3,y^3)$, and $D[x,y]/(x^4,y^3)$ respectively.

Examples

D[x,y]/(x^3,y^3)



$D[x,y]/(x^4,y^3)$

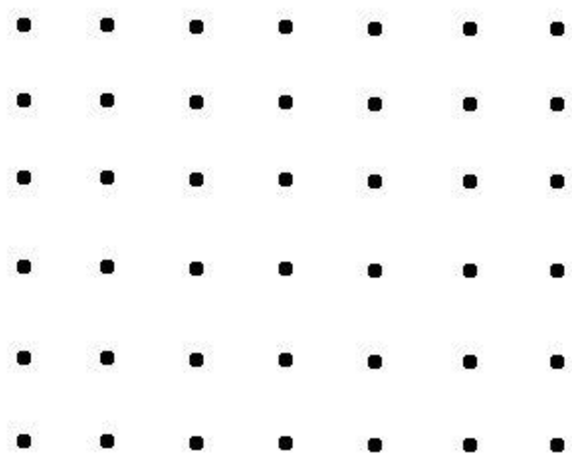

[x^3y]
[x^2y^2]
[x^2y]
[x^3y^2]
[xy^2]
[x,y]
[xy]
[x,y^2]
[y^2]
[x^2,y]
[y]
[x^2,y^2]
[x^3]
[x^3,y]
[x^2]
[x^3,y^2]
[x]
[x^2y,y^2]
[xy^2,x^2y,x^3]
[x^3y,y^2]
[y^2,xy,x^3]
[xy^2,x^2]
[y^2,xy,x^2]
[x^3,xy^2]
[xy^2,x^3y]
[x^2y,y^2x]
[x^2y^2,x^3y]
[x^3y,y^2x]
[x^3,x^2y^2]
[x^2,xy]
[x^3,xy]
[x^3,x^2y]

Remember?

In the last presentation of this material, we saw the outline of the proofs concerning the previous graphs. Among them was uniqueness in 2 dimensions, and that there was a one to one correspondence between the antichains and the nodes of the graph. We move on to more general rings now.

New Material

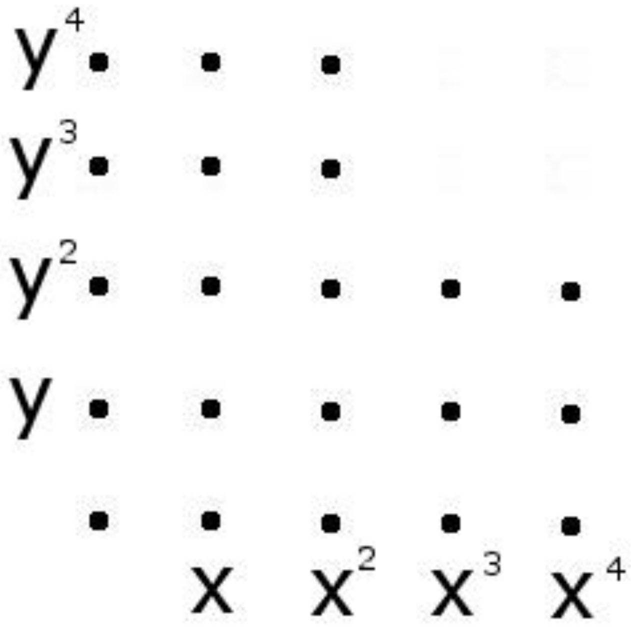
So far we have only seen monomial ideals consisting of the each variable raised to a power. These monomials lead to convex monomial lattices. For example:



New Material

If we consider a monomial ideal with a mixed monomial in it, we get a much different picture though. Consider the ring $R=D[X,Y]/(x^5,x^3y^3,y^5)$. The resulting monomial lattice is the concave shape:

New Material



New Material

The proximity of the points x^2y^4 and x^4y^2 causes $Ann(x^2y^4)=P(M_I)-\{1\}=Ann(x^4y^2)$.

The main consequence of this is that we lose the one to one correspondence between the antichains and the nodes of the graph when we add mixed monomials to the monomial ideal.

New Material

Definition: In $R = D[x_1,\dots, x_n]/I$, where I is a monomial ideal such that each x_i^k exists as one of the generators of I. We say a *corner* is any monomial, X, such that $x_iX=0$ for all i. As such, $Ann(X) = P(M_I) - \{1\}$.

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This leads us to the following lemma.

New Material

Lemma: Suppose $R = D[x_1, \ldots, x_n]/I$, with I as before, then in $\Gamma_a(R)$, the antichains made from the set of the corners are represented by the vertex with maximal degree.

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Lemma: Suppose $R = D[x_1,\dots, x_n]/I$, with I as before, then in $\Gamma_a(R)$, the antichains made from the set of the corners are represented by the vertex with maximal degree. Furthermore, for X a corner, and x_k the variable of highest degree in X, if $X=x_kX_k$, then X_k is the monomial whose vertex has the second highest degree in $\Gamma_a(R)$.

New Material

Proof outline: The annihilator of a corner is $P(M_l) - \{1\}$, and as such, the degree of the vertex represented by a corner is $|\Gamma_a(R)|$.

For the second half of the lemma, we need to look at the monomial X_k. We prove this is in fact the second highest degree by looking at how much less than the maximal degree it is.

Proof Outline (cont.)

We can't calculate directly how much less it is, but we can describe it in terms of antichains. If deg(X)=D, then $deg(X_k)=D-D_k$ for some D_k. The way X_k is defined, we can see that D_k is in fact exactly the number of antichains in M_l which do not contain the variable x_k. Since x_k is maximal, this minimizes D_k.

Theorem

Suppose $R_1 = D[x_1,\ldots, x_n]/I$, and $R_2 = D[x_1,\ldots, x_n]/J$ where I and J are monomial ideals with each variable to a power as one of the generators. Then $R_1 \cong R_2$ if and only if $\Gamma_a(R_1)$? $\Gamma_a(R_2)$.

Proof

The proof of this theorem follows directly from the previous lemma.

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Proof

The lemma tells us formally what that degree is. To compare the vertices from the different graphs, we need only look at the size of the respective D_k's as defined in the lemma.

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As always, further details can be provided.

Not quite finished

There is one noticeable case missing here, and that is the case where I is a monomial ideal in n variables that has at least one variable that is not an individual generator. This is missing for the following reason:

Woops

Let $R_1=D[X,Y]/(x^3,x^2y^2)$, and $R_2=D[X,Y]/(x^3,x^2y^2,y^3)$, then $\Gamma_a(R_1) \cong \Gamma_a(R_2)$ when clearly the rings are not isomorphic.

This isn’t all bad though. We are lead to a possible new theorem:

New Theorem?

Definition: The Krull dimension of a ring is the number of strict inclusions in a maximal chain of prime ideals.

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Conjecture: Let $\mathcal{R}=\{R \mid R= D[x_1,\ldots, x_n]/I$, I any monomial ideal$\}$. Let $\mathcal{R}_k=\{R \in \mathcal{R} \mid R$ has Krull dimension k$\}$, then for all $R_i \in \mathcal{R}_k$, $R_1 \cong R_2$ if and only if $\Gamma_a(R_1) \cong \Gamma_a(R_2)$.