Subgroup Lattices and their Chromatic Number

Voula Collins

Missouri State University REU

Summer 2008

Groups and Subgroups

A group is a set of elements with a binary operation that satisfy the properties

- Closure
- Associativity
- Identity
- Inverse

A subgroup is a subset of a group such that the same four properties hold.

Subgroup Lattices

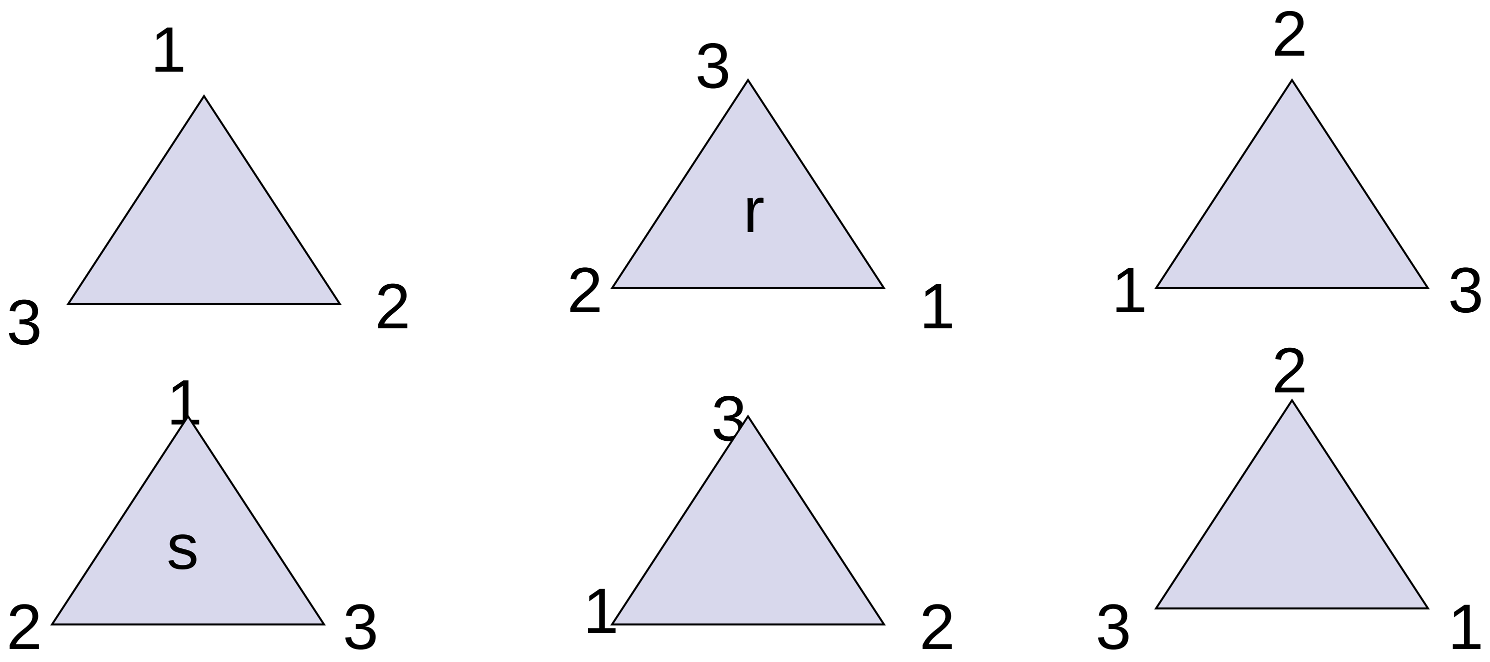
A subgroup lattice is a graph associated with a group such that

- vertices are the subgroups of G
- an edge connects vertices M and N if M≤N and there is no intermediate subgroup(or vice versa)

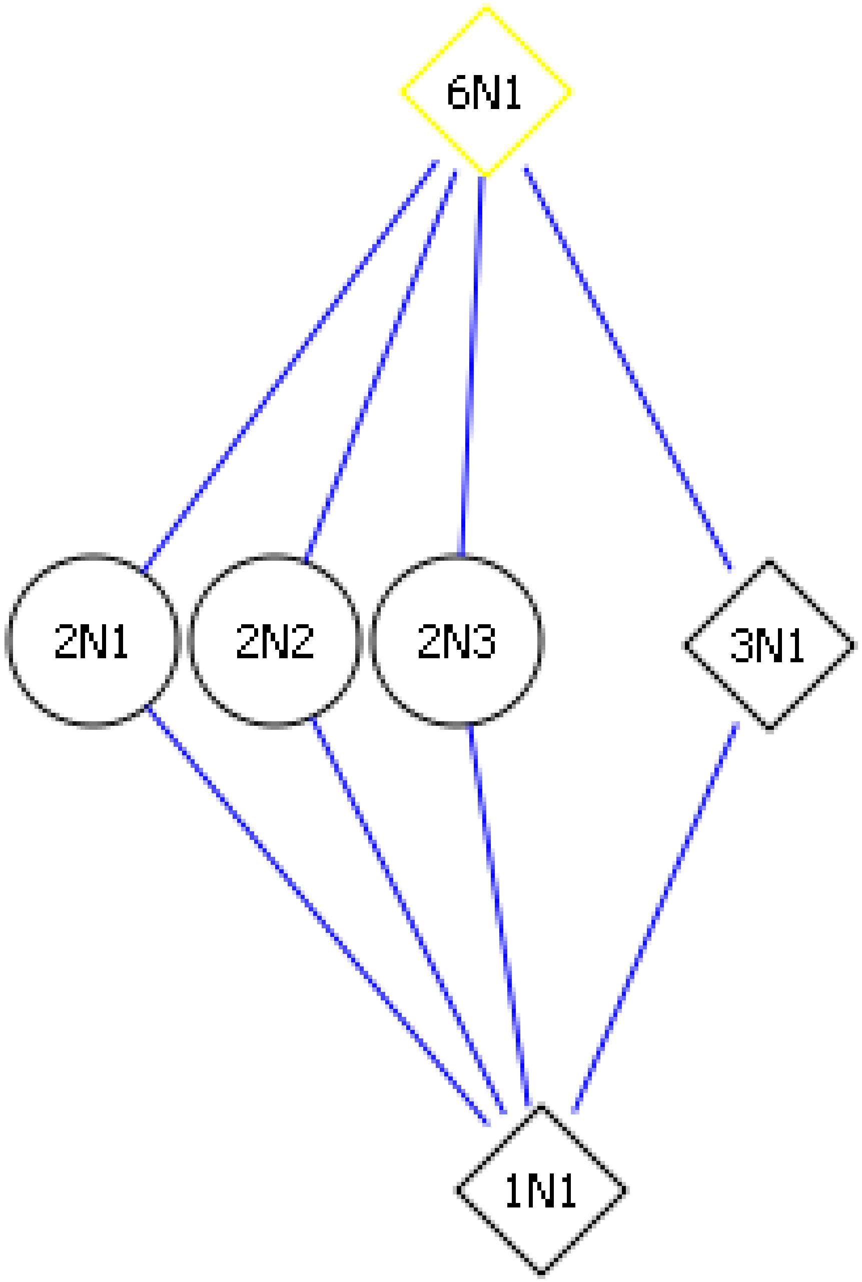
Example: D_6

- The symmetries of an equilateral triangle.
- generators and relations:

$$< r,s \mid r^3 = 1, s^2 = 1, srs^{-1} = r^{-1} >$$



D_6

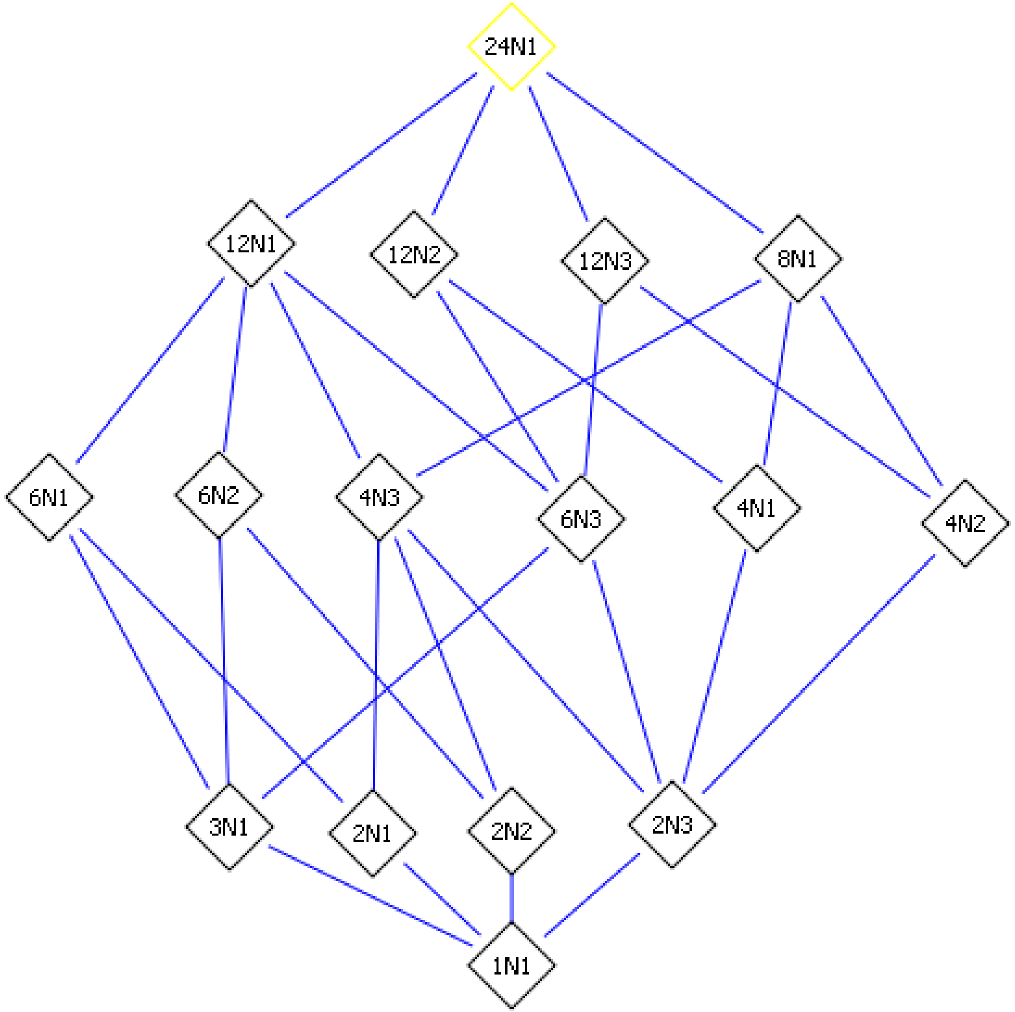


Chromatic Number

- The chromatic number of a graph is the minimum number of colors one can use to color the vertices of the graph so that no two adjacent vertices are the same color.
- If the chromatic number of a graph is two, then it is called bipartite.

Abelian Groups

$\mathbb{Z}_4 \times \mathbb{Z}_6$



Other Bipartite Groups

- Abelian groups are bipartite

- P-groups are bipartite

- Cyclic semidirect cyclic groups are bipartite
  - Dihedral groups are in this category

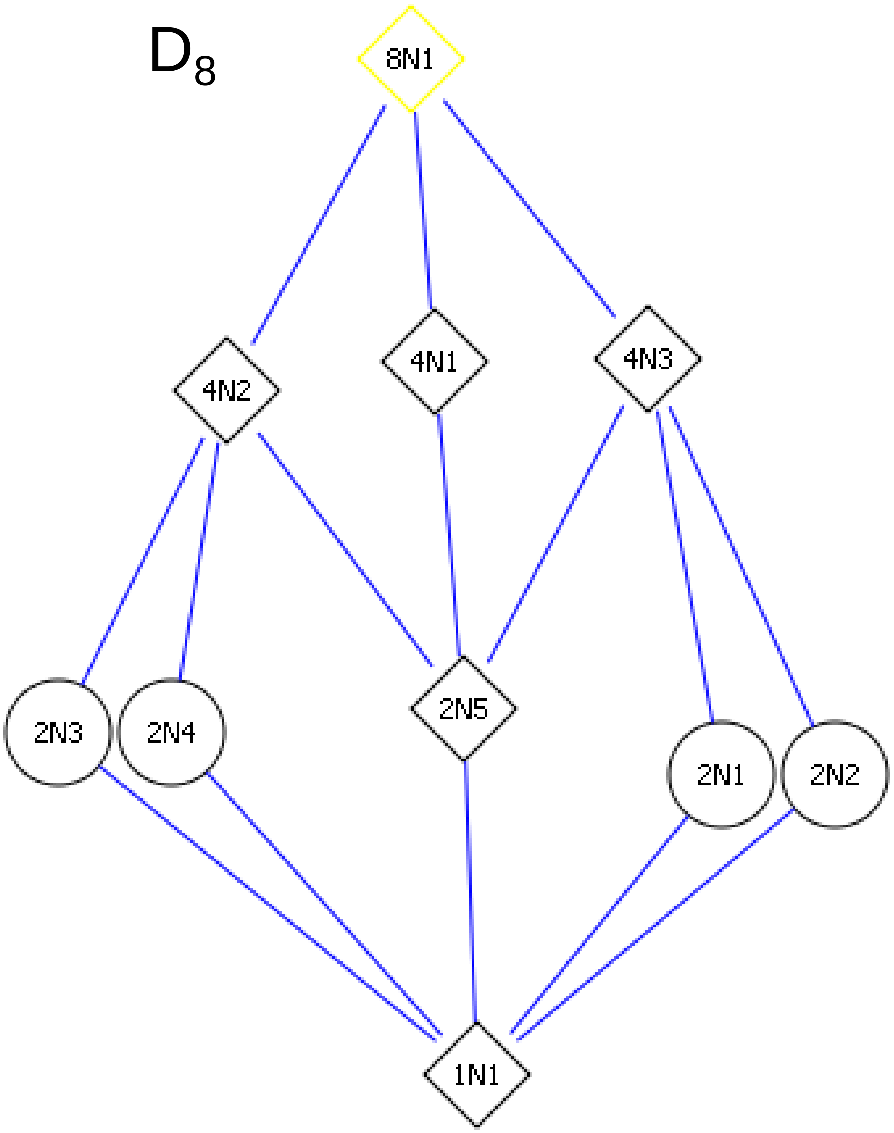
$$\mathbb{Z}_n \rtimes \mathbb{Z}_m$$

Tying it All Together

All of the groups mentioned in the previous slide have the property of being supersolvable, which give them a very regular structure.

A subgroup lattice is Dedekind-Jordan if every upward path from the trivial group to the entire group through the lattice is the same length.

D_8



Kenkichi Iwasawa proved that a subgroup lattice of a group is Dedekind iff the group is supersolvable.

It is easy to see that a lattice is bipartite if it is Dedekind

However there are bipartite lattices which aren’t Dedekind.

Other Subgroup Lattices

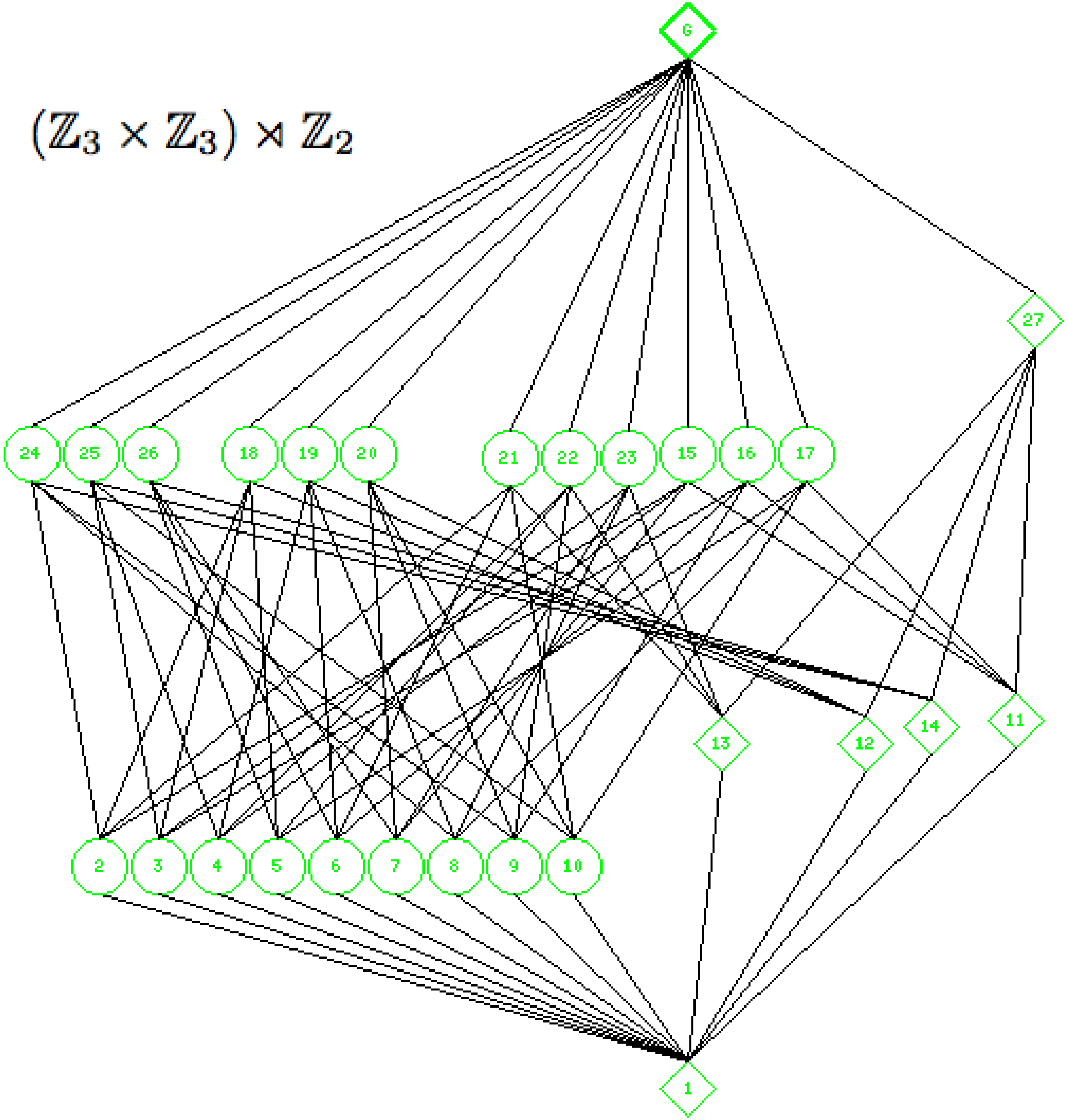
Another collection of subgroup lattices we have been investigating are of the form

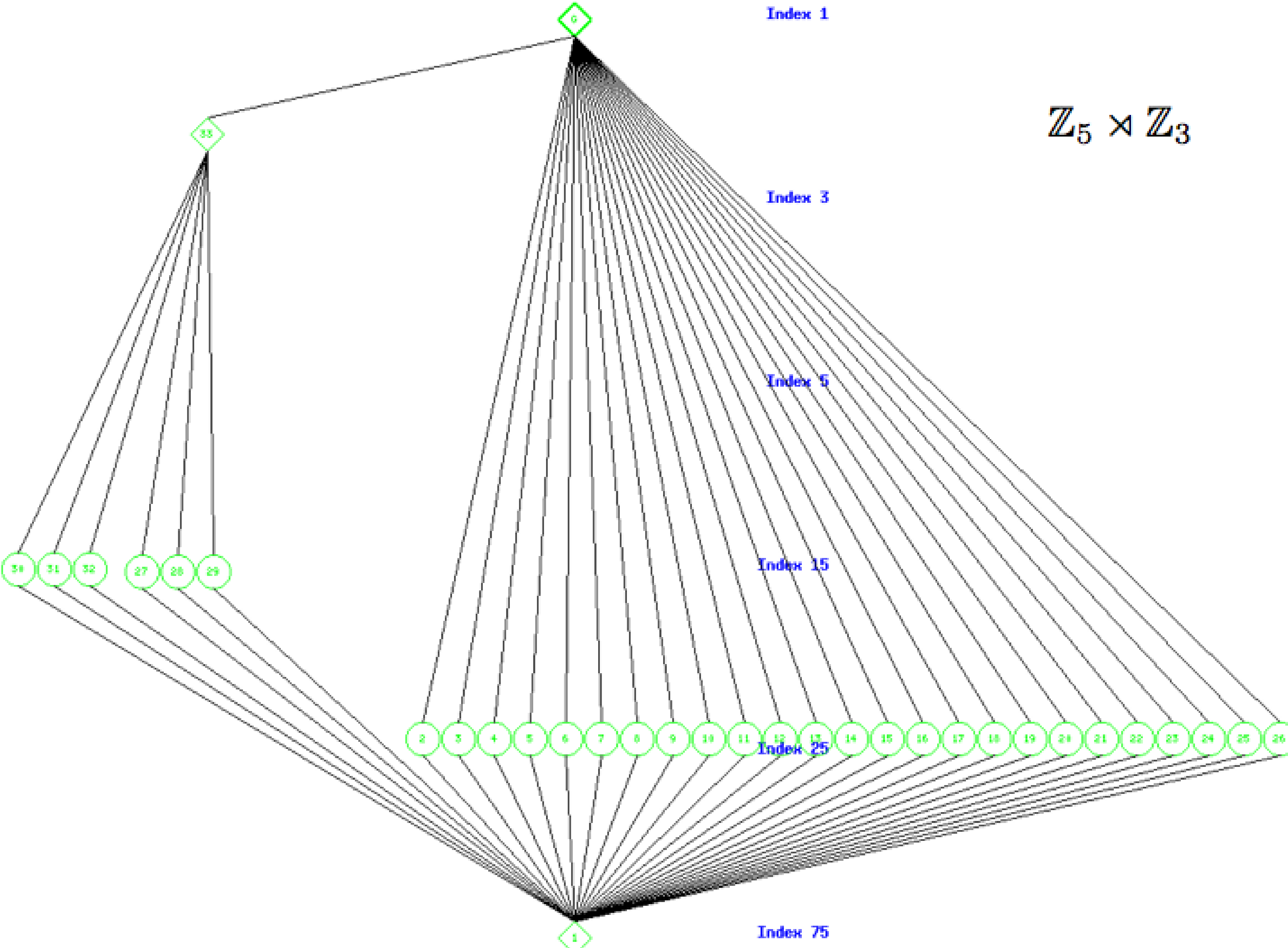
$$(\mathbb{Z}_p)^k \rtimes \mathbb{Z}_n$$

We have shown that these groups are supersolvable, and thus bipartite, when n|p-1.

There are examples of tripartite lattices when n|p+1 and non-dedekind bipartite lattices when $n|p^2+p+1$ where n is prime.

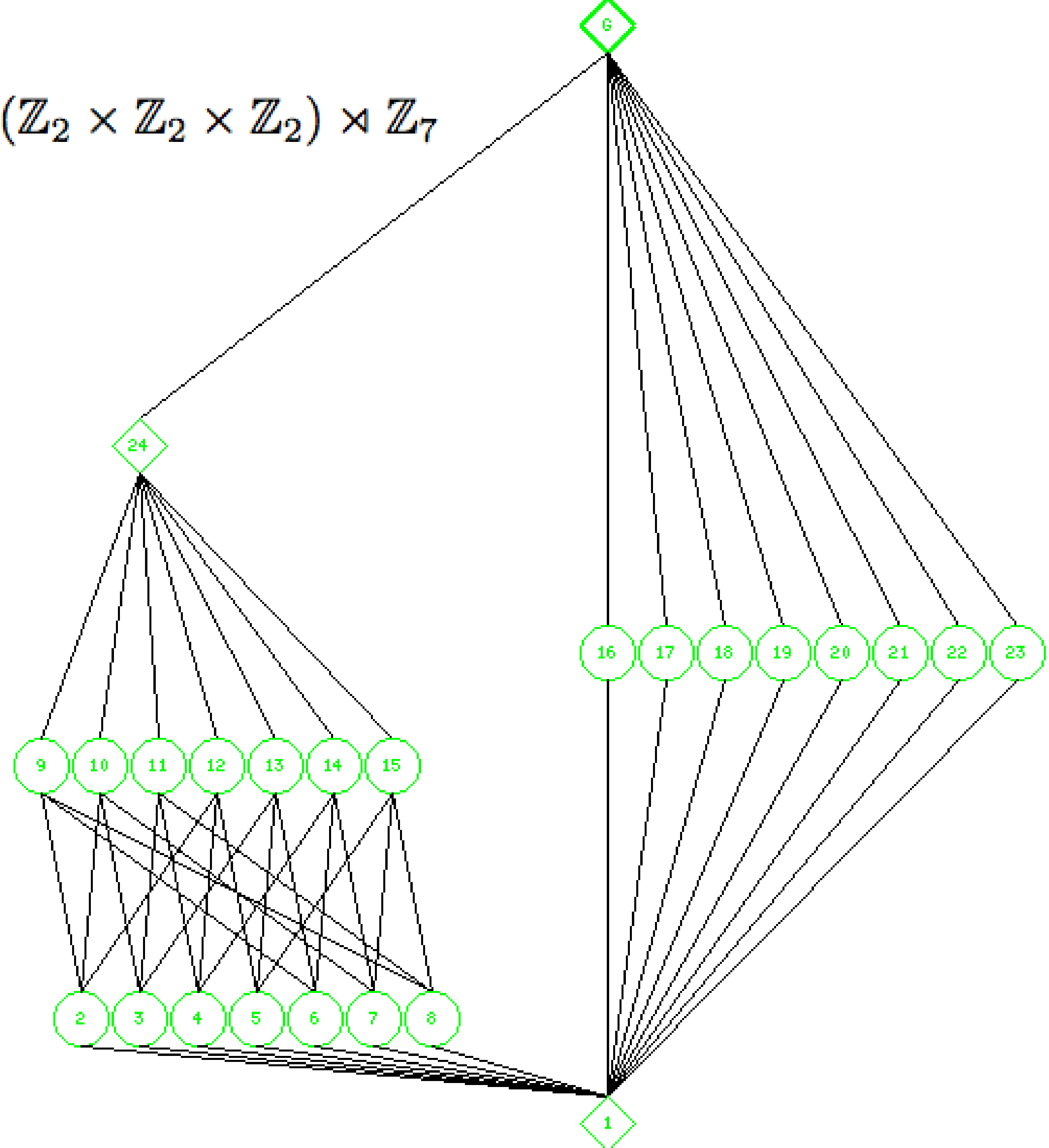
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$





Index 1
$\mathbb{Z}_5 \rtimes \mathbb{Z}_3$
Index 3
Index 5
Index 15
Index 25
Index 75

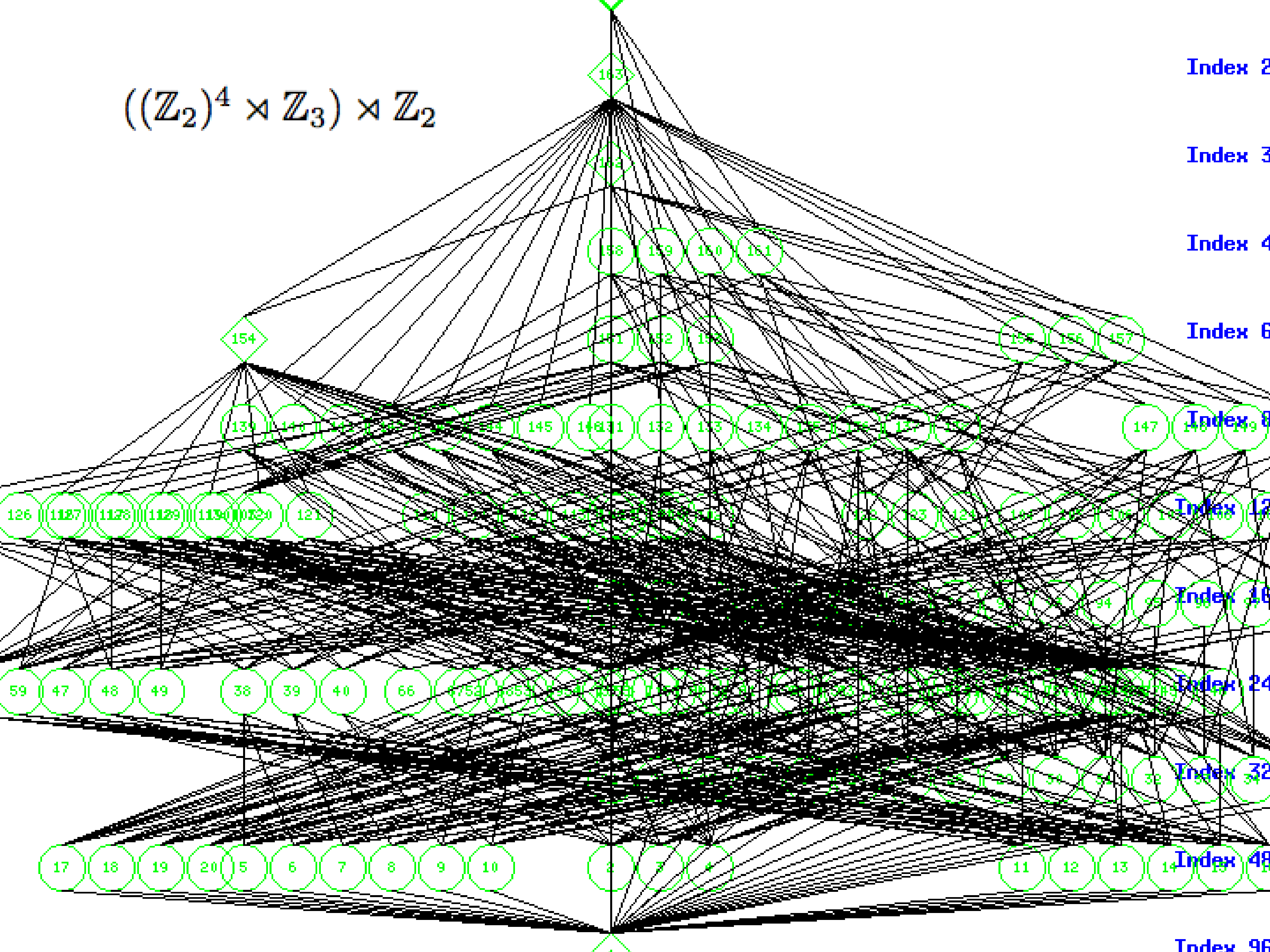
$(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_7$



Chromatic Number Four

One question of interest is whether the chromatic number of lattices, increases arbitrarily. We begin by attempting to find a any lattice with chromatic number four.

- Exhaustive search of subgroup lattices
- Construction

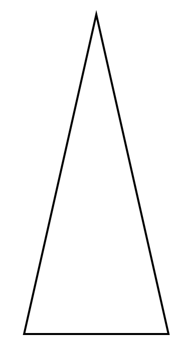

$((\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$
Index 2
Index 3
Index 4
Index 6
Index 8
Index 12
Index 16
Index 24
Index 32
Index 48
Index 96

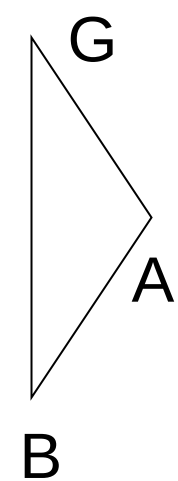
Conjugacy Classes

- A way of simplifying the graphs we get in GAP is to instead consider the coloring of the conjugacy class lattice.
- This conjugacy class lattice gives a lower bound for the chromatic number of the subgroup lattice

Construction

- Subgroup lattices are triangle free.
- There are ways of constructing triangle free graphs with high chromatic number(i.e. Mycielski's construction), and we hope to use similar methods to construct lattices with larger chromatic number as well.

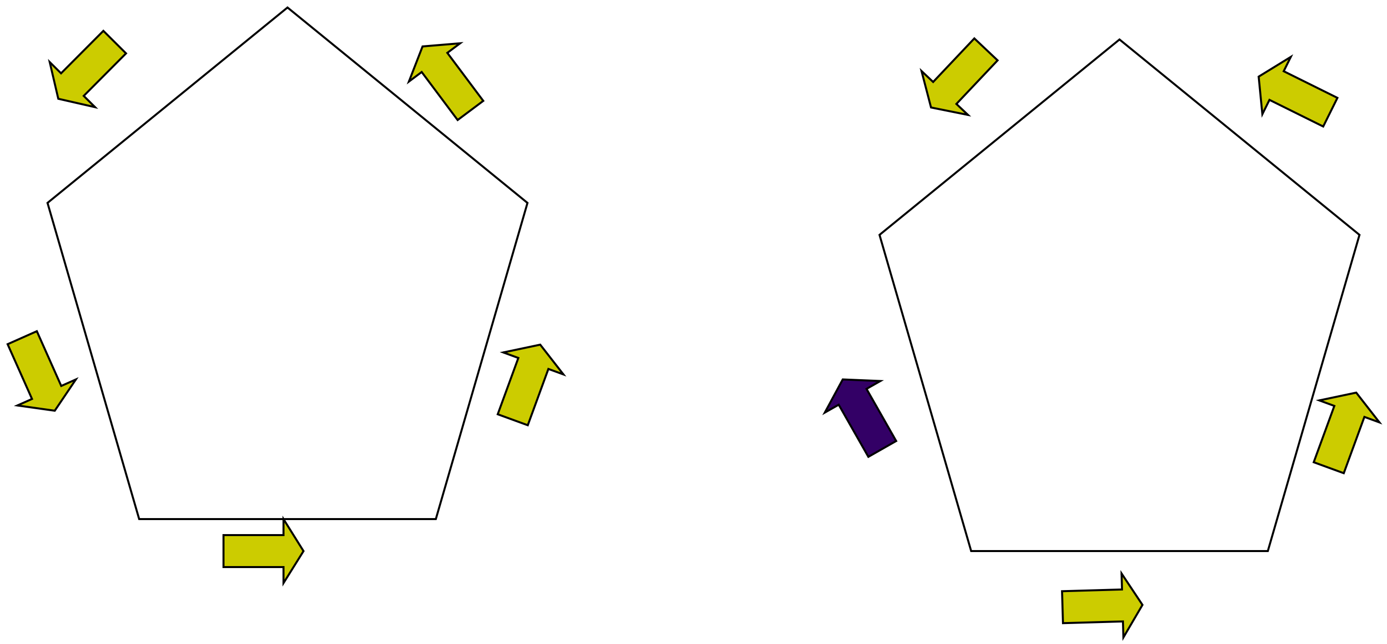




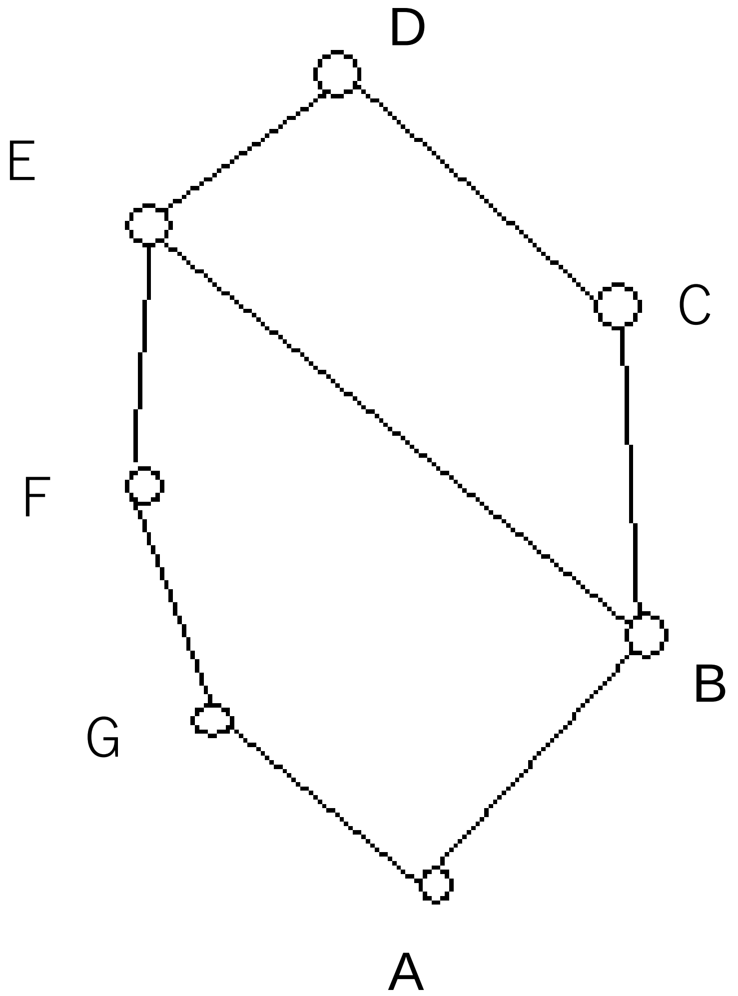
Lattices and Digraphs

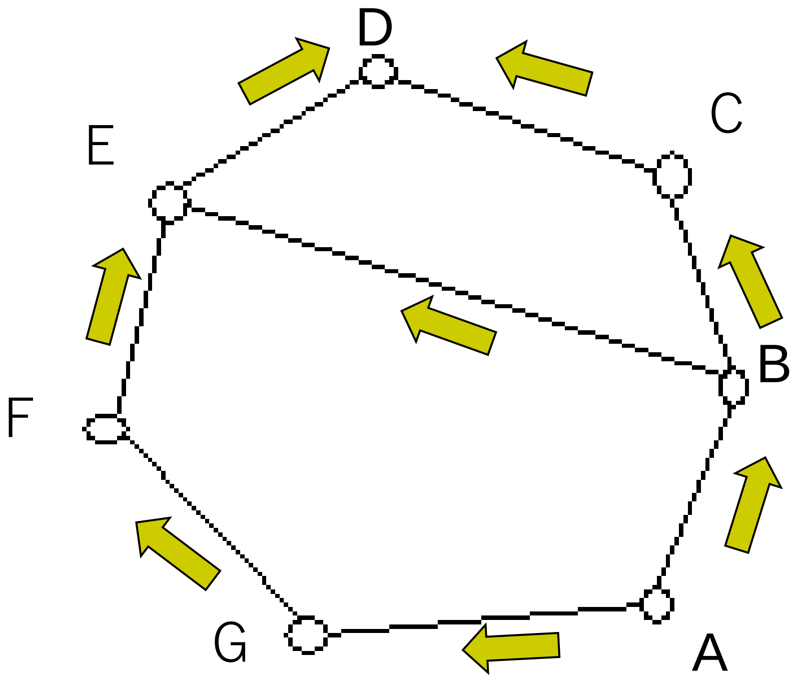
- Lattices can be represented as directed graphs, I.e. graphs where edges have a direction.
- Here the direction represents which way is going up the lattice
- Therefore there can be no cycles or “shortcuts”

Cycles and Shortcuts

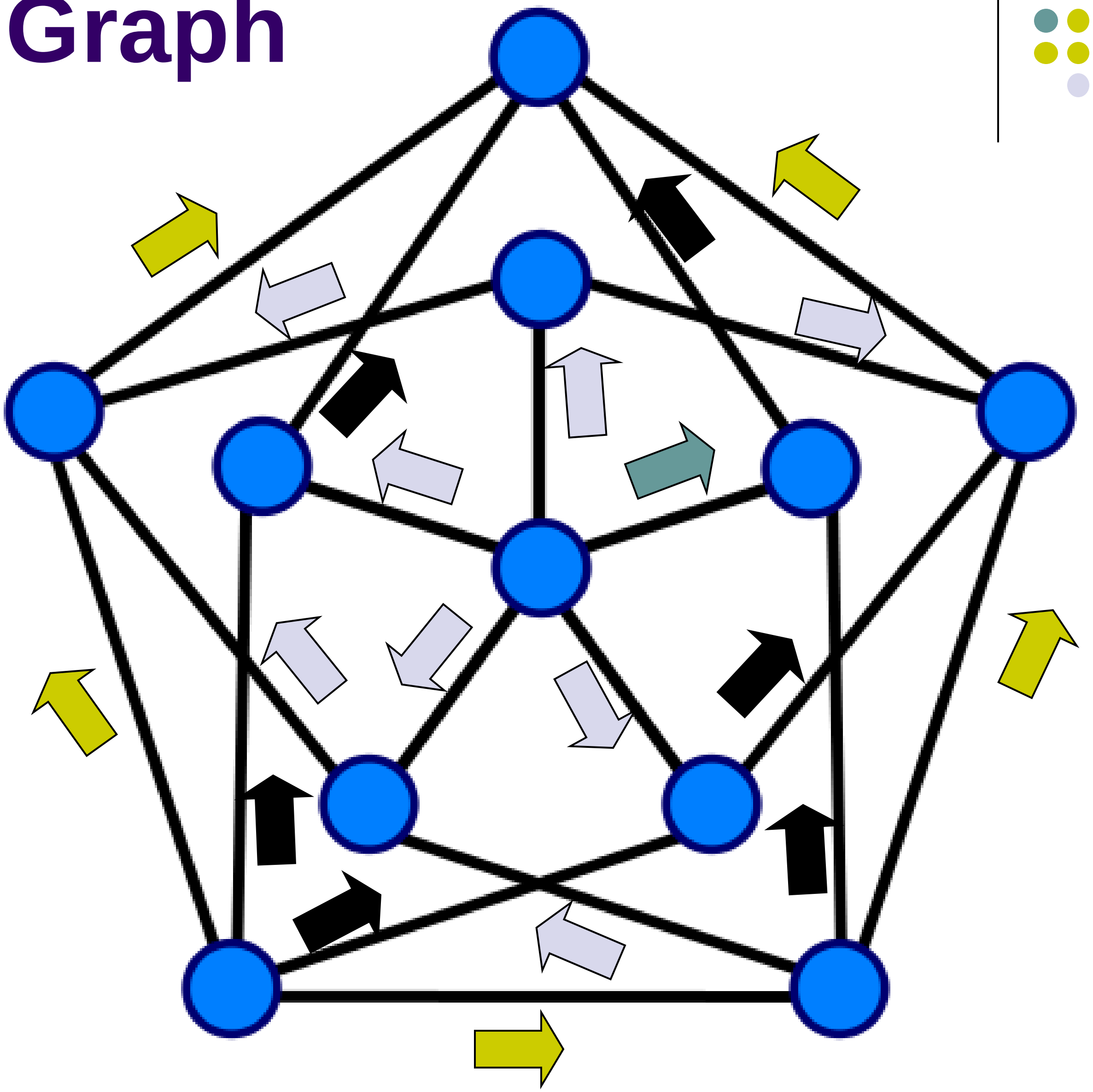


Example





Grotzsch Graph



Subgroup Lattices of Infinite Abelian Groups

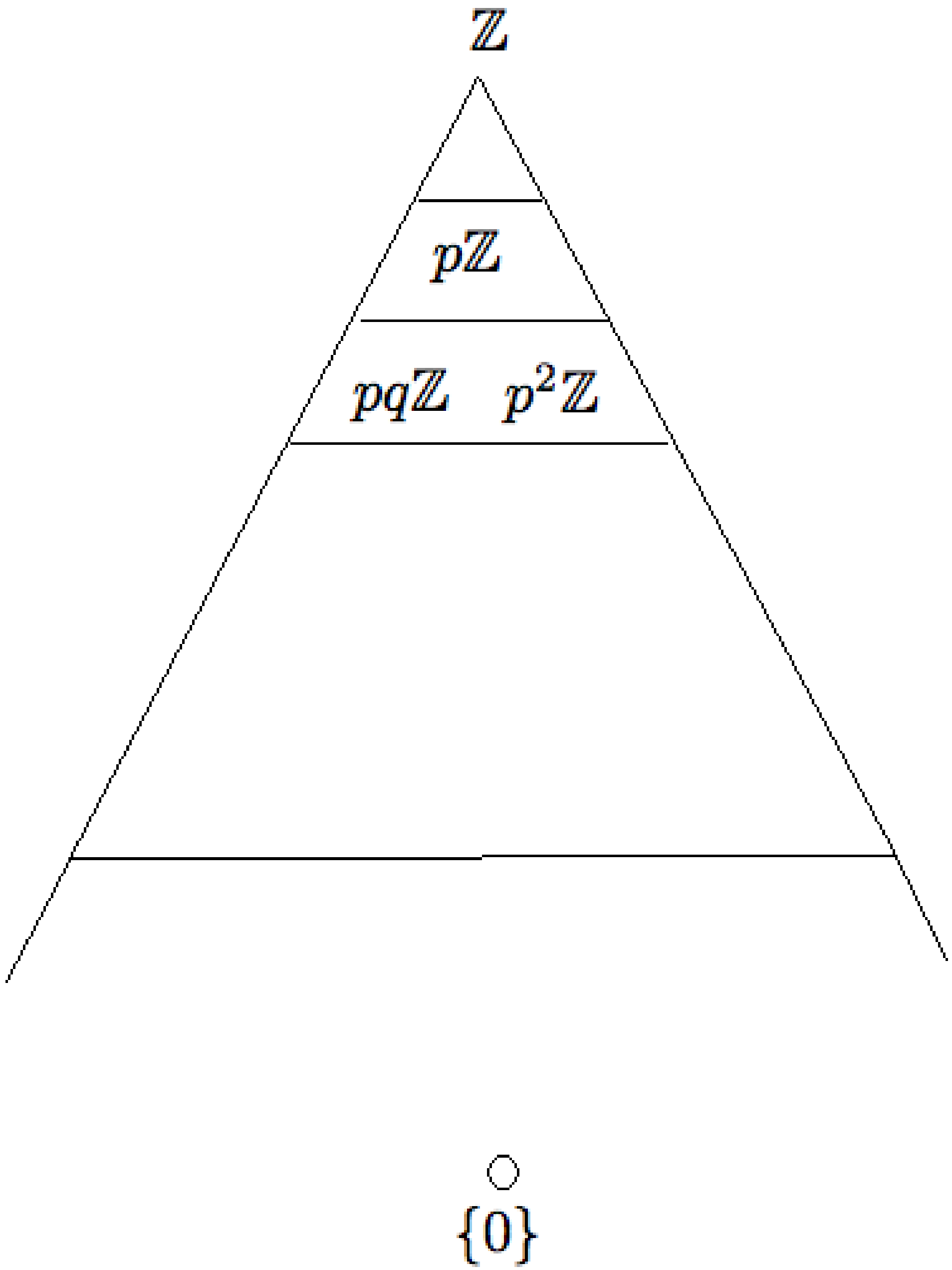
Finitely Generated Abelian Groups

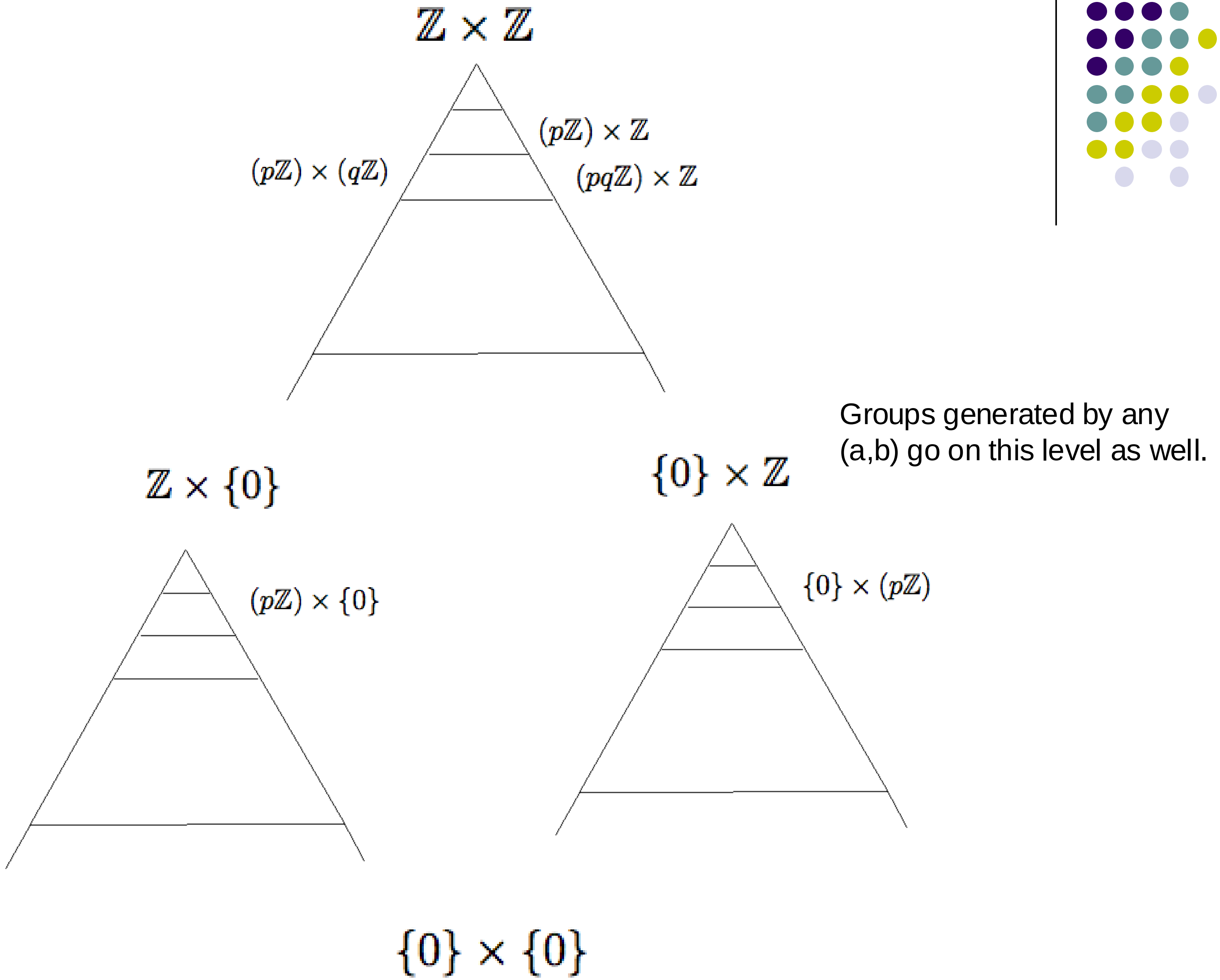
- Finitely generated of abelian groups are of the form

$$(\mathbb{Z})^k \times A$$

  where A is finite abelian.

- All finitely generated abelian groups can be shown to be bipartite.


$\mathbb{Z}$
$p\mathbb{Z}$
$pq\mathbb{Z}$ $p^2\mathbb{Z}$
$\{0\}$


$\mathbb{Z} \times \mathbb{Z}$
$(p\mathbb{Z}) \times \mathbb{Z}$
$(p\mathbb{Z}) \times (q\mathbb{Z})$
$(pq\mathbb{Z}) \times \mathbb{Z}$
Groups generated by any (a,b) go on this level as well.
$\mathbb{Z} \times \{0\}$
$\{0\} \times \mathbb{Z}$
$(p\mathbb{Z}) \times \{0\}$
$\{0\} \times (p\mathbb{Z})$
$\{0\} \times \{0\}$

Infinitely Generated Abelian Groups

- There is no such general form for for infinitely generated abelian group.
- Examples:

$$\mathbb{Q}$$

$$\mathbb{Z}_{p^\infty}$$

where $\mathbb{Z}_{p^k}$ gives the p^kth complex roots of one.

Let N(x) be the number of non-distinct prime divisors of x.

$\mathbb{Q}$

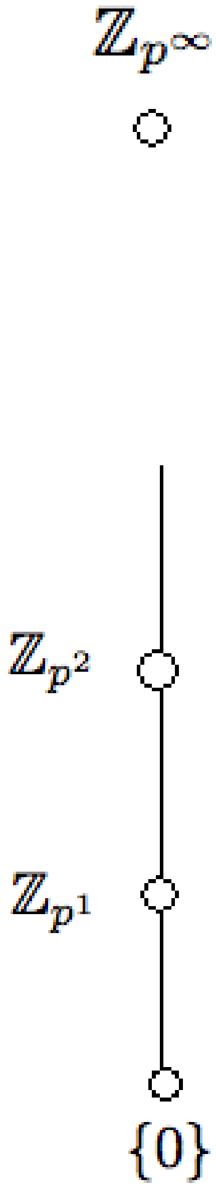
$\frac{a}{b}\mathbb{Z}$ Where N(a)-N(b) = -1

$\frac{a}{b}\mathbb{Z}$ Where N(a)-N(b) = 0

$\frac{a}{b}\mathbb{Z}$ Where N(a)-N(b) = 1

$\frac{a}{b}\mathbb{Z}$ Where N(a)-N(b) = 2

$\{0\}$



$\mathbb{Z}_{p^\infty}$
$\mathbb{Z}_{p^2}$
$\mathbb{Z}_{p^1}$
$\{0\}$

Future Goals

- Further investigate infinite groups, abelian and non-abelian.
- Fill in the gaps for our finite semi-direct products.
- Prove one way or the other for the existence of chromatic number four lattices and subgroup lattices.