A Spectral Analysis of Cyclic and Elementary Abelian Subgroup Lattices

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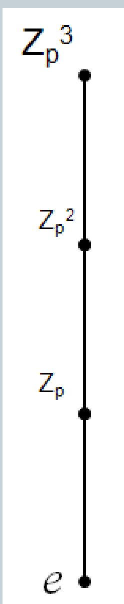
Background Material

- Recall the definitions of:
  - Group
  - Graph
  - Subgroup Lattice – A graph where each subgroup is a vertex and two subgroups are connected iff one is a subgroup of the other without any intervening subgroups.
  - Direct Product
  - Adjacency Matrix – A square, symmetric matrix which represents a graph by placing a 1 in the i-th row and j-th column if i ~ j and a 0 elsewhere.
  - Characteristic Polynomial
  - Eigenvalue
  - Spectrum – The set of eigenvalues of the adjacency matrix of a graph

Background Material

- Recall: Group → Subgroup graph → Adjacency matrix → Characteristic polynomial → Eigenvalues
- For example:



→

$$\begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = \lambda^4 - 3\lambda^2 + 1 = 0$$

↓

$$\pm\phi \text{ and } \pm\frac{1}{\phi}$$

Graph Products

- A *graph product* is a binary operation, denoted "□", on graphs which induces a new graph over the Cartesian product of the graph vertices:

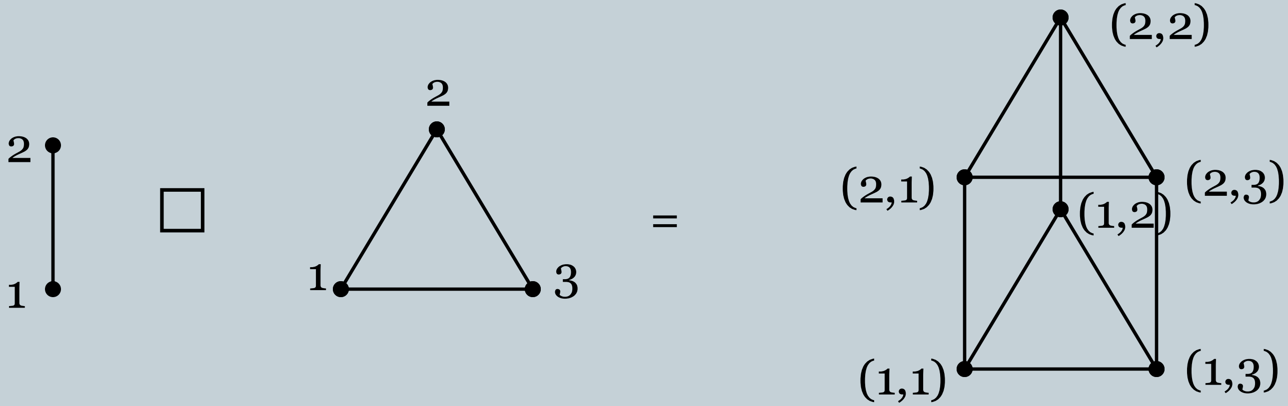
$$G \ \mathrm{W} H = (V(G) \times V(H), E(V(G) \times V(H)))$$

where $((g_1, h_1), (g_2, h_2)) \in E(V(G) \times V(H))$ iff

$$((g_1 = g_2) \wedge (h_1 \sim h_2)) \vee ((g_1 \sim g_2) \wedge (h_1 = h_2))$$

Graph Products (Cont'd)

- For example:



- NB: When one of the graphs is a path graph, such as here, the graph product has the visual effect of "projecting" the other graph into the next dimension.

Cyclic Groups

- A *cyclic group* is a group such that all elements of the group can be expressed as a power of a single element of the group, called a *generator*, of which there may be more than one in the group.
- Every cyclic group of order n is isomorphic to (essentially the same as) a group of the form {0, 1, 2, 3…n-1} under addition modulo n, denoted $\mathbf{Z}_n$.

Cyclic Groups (Cont'd)

- Rewriting $\mathbf{Z}_n$:

$$\mathbf{Z}_{p_1^{\alpha_1}} \, p_2^{\alpha_2} \, p_3^{\alpha_3} \cdots p_s^{\alpha_s}$$

where

$$n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_s^{\alpha_s}$$

- A group of the form $\mathbf{Z}_{p^\alpha}$ is called a *p-group*.

Cyclic Groups (Cont'd)

- Theorem:

$$\mathbf{Z}_n \cong \bigoplus_{i=1}^{s} \mathbf{Z}_{p_i^{\alpha_i}}$$

- That is, every cyclic group is the direct product of p-groups where the orders of the p-groups are the prime-power factors of the order of the cyclic group.

Graph Product – Direct Product

- Theorem:

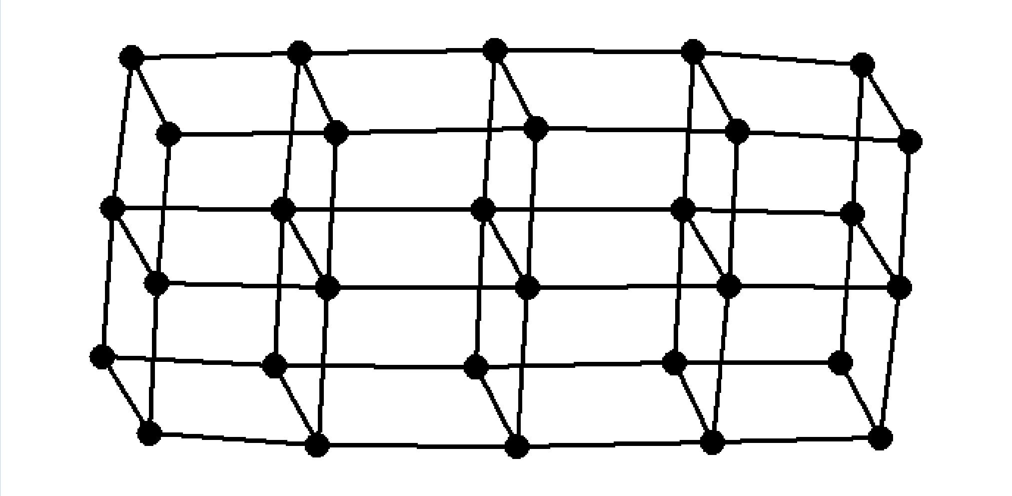
$$\Gamma(G \oplus H) \cong \Gamma(G)\ \mathrm{W}\Gamma(H) \text{ if } \gcd(|G|,|H|) = 1$$

- That is, the subgroup lattice of a direct product is equal to the graph product of the component subgroup lattices if the groups are coprime.
- Because the component p-groups of a cyclic group are always pairwise coprime, the subgroup lattice of every cyclic group is the graph product of all its component p-group subgroup lattices.

Graph Product – Direct Product (Cont'd)

- Because the subgroup lattice of every p-group is a path graph, this means that the subgroup lattice of every cyclic group is a simple cubic lattice in *n* dimensions, where n is the number of distinct p-groups which produce the cyclic group.

$$\mathbf{Z}_{11250} \cong \mathbf{Z}_{2^1} \oplus \mathbf{Z}_{3^2} \oplus \mathbf{Z}_{5^4} \longrightarrow$$



Graph Product – Adjacency Matrix

- Taking a graph product has a predictably regular effect on the adjacency matrix of the product graph:

$$A_3 \, \mathrm{W} B_3 = \begin{pmatrix} A & b_{1,2} I_3 & b_{1,3} I_3 \\ b_{2,1} I_3 & A & b_{2,3} I_3 \\ b_{3,1} I_3 & b_{3,2} I_3 & A \end{pmatrix}$$

- That is, the adjacency matrix of the product graph is a *block matrix* composed of one of the adjacency matrices along the diagonal and identities of equal dimension where the other matrix' entries equal 1.

Tridiagonal Block Matrices

- Because p-group adjacency matrices are always tridiagonal, (non-zero entries are restricted to the diagonal, superdiagonal, and subdiagonal) the adjacency matrix of arbitrarily many p-groups is always a tridiagonal block matrix.
- Thus, every cyclic group adjacency matrix has a tridiagonal block form.
- There is a formula to give the determinant of such matrices. [Molinari, 2008]

Prime Result

- Theorem: Given any cyclic group, $\mathbf{Z}_n \cong \bigoplus_{i=1}^{s} \mathbf{Z}_{p_i^{\alpha_i}}$, its eigenvalues are

$$\sum_{i=1}^{s} \{2\cos(\frac{j\pi}{\alpha_i + 1})\}_{j=0}^{\alpha_i}$$

- That is, its spectrum is the Cartesian sum of the spectra of all its component p-groups.
- This settles the matter for all cyclic groups.

Elementary Abelian Groups

- An *elementary Abelian group* is a group of the form

$$(\mathbf{Z}_p)^n = \bigoplus_{i=1}^{n} \mathbf{Z}_p$$

- These are interesting because $\mathbf{Z}_p$ is a field, which means that $(\mathbf{Z}_p)^n$ is a finite vector space where subgroups correspond to subspaces. This allows us to more easily determine the subgroup lattice structure of these groups and also their spectra.

Elementary Abelian Groups (Cont'd)

- Elementary Abelian groups are also very interesting because they are the first case of the larger family of groups, $(\mathbf{Z}_{p^n})^m$, the carefully chosen subgroups of which, for distinct *p*, comprise all the coprime factors of any Abelian group.
- Results:

$$\left|A_{(\mathbf{Z}_p)^3}\right| = (p-\lambda^2)^{(p^2+p)}(2\lambda^4 - 7p^2\lambda^2 - 6p\lambda^2 - 3\lambda^2 + 4p^4 + 3p^3 - 4p^2 - 9p - 4)$$

- This gives the spectrum for all groups of the form $(\mathbf{Z}_p)^3$
- These matrices are also tridiagonal in general, so this is promising.

What's Next

- Continue exploring block matrix methods in solving all elementary Abelian groups.
- Expand upon these methods to solve all Abelian groups.
- Solve dihedral groups and other semidirect products.
- What can be said about non-isomorphic groups with cospectral subgroup lattices?
- Spectra of subring lattices?
- And for something completely different: what groups have subgroup lattices of genus 1 (can be drawn on a torus without intersection)?