# A Spectral Analysis of Cyclic and Elementary Abelian Subgroup Lattices

#### **C H R I S T O P H E R M . P . T O M A S Z E W S K I**

**3 0 J U L Y M M I X**

# Background Material

#### Recall the definitions of:

- Group
- Graph
- $\circ$  Subgroup Lattice A graph where each subgroup is a vertex and two subgroups are connected iff one is a subgroup of the other without any intervening subgroups.

#### Direct Product

- Adjacency Matrix A square, symmetric matrix which represents a graph by placing a 1 in the i-th row and j-th column if  $i \sim j$  and a 0 elsewhere.
- Characteristic Polynomial
- o Eigenvalue
- Spectrum The set of eigenvalues of the adjacency matrix of a graph

## Background Material

- Recall: Group  $\longrightarrow$  Subgroup graph  $\longrightarrow$  Adjacency  $matrix \longrightarrow$  Characteristic polynomial  $\longrightarrow$  Eigenvalues
- For example:



#### Graph Products

 A *graph product* is a binary operation, denoted "□", on graphs which induces a new graph over the Cartesian product of the graph vertices: *Graph Products*<br> *A graph product is a binary operation, denoted "a",*<br>
on graphs which induces a new graph over the<br> *Green Cartesian product of the graph vertices:*<br> *GWH = (V(G)×V(H), E(V(G)×V(H))*)<br>
where  $((g_1, h_1), ($ 

where  $((g_1, h_1), (g_2, h_2)) \in E(V(G) \times V(H))$  iff

$$
((g_1 = g_2) \wedge (h_1 \sim h_2)) \vee ((g_1 \sim g_2) \wedge (h_1 = h_2))
$$



• NB: When one of the graphs is a path graph, such as here, the graph product has the visual effect of "projecting" the other graph into the next dimension.

## Cyclic Groups

- A *cyclic group* is a group such that all elements of the group can be expressed as a power of a single element of the group, called a *generator*, of which there may be more than one in the group.
- Every cyclic group of order *n* is isomorphic to (essentially the same as) a group of the form {0, 1, 2, 3... $n-1$ } under addition modulo  $n$ , denoted  $\mathbb{Z}_n^n$ .

#### Cyclic Groups (Cont'd)



Toups (Compared to  $\bigcirc$  1  $p_2^{\alpha_2} p_3^{\alpha_3}$ L  $p_s^{\alpha}$ Groups (C<br>
Troups (C<br>  $\frac{a_1}{1} p_2^{\alpha_2} p_3^{\alpha_3} L p_3^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} L p_3^{\alpha_4}$ *s Groups* (Cont'd<br> **n**<br>  $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} L p_s^{\alpha_s}$ <br>  $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} L p_s^{\alpha_s}$ Groups (Cont'd)<br>  $Q$ <br>  $\frac{a_1}{p_2^{\alpha_2}p_3^{\alpha_3}L}$   $p_s^{\alpha_s}$ ic Groups (Co<br>  $\sum_{p_1}$ <br>  $\sum_{p_1}$ <br>  $\sum_{p_2}$ <br>  $p_3^{\alpha_1}$ <br>  $p_2^{\alpha_2}$ <br>  $p_3^{\alpha_3}$ <br>  $\sum_{p_1}$ <br>  $p_2^{\alpha_4}$ <br>  $p_3^{\alpha_5}$ <br>  $\sum_{p_1}$ <br>  $\sum_{p_2}$ <br>  $\sum_{p_3}$ <br>  $\sum_{p_3}$ <br>  $\sum_{p_3}$ Froups (Compared to  $\bigcirc$ <br>  $\bigcirc$   $\bigcirc$   $\bigcirc$   $\bigcirc$   $\bigcirc$   $\bigcirc$   $p_3^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} L$  is called lic Groups (Cont'd)<br>  $\sum_{p_1}$ <br>  $\sum_{p_1}$ <br>  $\sum_{p_2}$ <br>  $p_3$ <sup>*n*</sup><br>  $p_s$ <br>  $p_s$ 

where

$$
n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} L p_s^{\alpha s}
$$

• A group of the form  $\mathbb{Z}_n$  is called a *p-group*. *p* **Zn**

## Cyclic Groups (Cont'd)

• Theorem:

#### $-1$   $p_i$ *i i s*  $n = \bigcup_{i=1}^{\infty} P_i^{\alpha_i}$

 That is, every cyclic group is the direct product of pgroups where the orders of the p-groups are the prime-power factors of the order of the cyclic group. velic Groups (Cont'd)<br>  $\sum_{n} \cong \bigoplus_{i=1}^{s} \sum_{p_i^{\alpha_i}}$ <br>
explic group is the direct product of p-<br>the orders of the p-groups are the<br>
actors of the order of the cyclic group.

#### Graph Product – Direct Product

#### • Theorem:

- That is, the subgroup lattice of a direct product is equal to the graph product of the component subgroup lattices if the groups are coprime.
- Because the component p-groups of a cyclic group are always pairwise coprime, the subgroup lattice of every cyclic group is the graph product of all its component p-group subgroup lattices. Graph Product – Direct Product<br>
Theorem:<br>  $\Gamma(G \oplus H) \cong \Gamma(G)$  W $\Gamma(H)$  if gcd( $|G|, |H|$ ) = 1<br>
That is, the subgroup lattice of a direct product is<br>
equal to the graph product of the component<br>
subgroup lattices if the groups

#### Graph Product – Direct Product (Cont'd)

 Because the subgroup lattice of every p-group is a path graph, this means that the subgroup lattice of every cyclic group is a simple cubic lattice in *n* dimensions, where n is the number of distinct pgroups which produce the cyclic group.



#### Graph Product – Adjacency Matrix

 Taking a graph product has a predictably regular effect on the adjacency matrix of the product graph:

raph Product – Adjacency Matrix  
\na graph product has a predictably reguli  
\nin the adjacency matrix of the product gr  
\n
$$
A_3 \text{ W}B_3 = \begin{pmatrix} A & b_{1,2}I_3 & b_{1,3}I_3 \\ b_{2,1}I_3 & A & b_{2,3}I_3 \\ b_{3,1}I_3 & b_{3,2}I_3 & A \end{pmatrix}
$$
\nthe adjacency matrix of the product gra  
\nmatrix composed of one of the adjacent  
\nes along the diagonal and identities of eq  
\nion where the other matrix' entries equa

• That is, the adjacency matrix of the product graph is a *block matrix* composed of one of the adjacency matrices along the diagonal and identities of equal dimension where the other matrix' entries equal 1. ph Product – Adjacency Matrix<br>graph product has a predictably regular<br>the adjacency matrix of the product gr<br>
3  $WB_3 = \begin{pmatrix} A & b_{1,2}I_3 & b_{1,3}I_3 \ b_{2,1}I_3 & A & b_{2,3}I_3 \ b_{3,1}I_3 & b_{3,2}I_3 & A \end{pmatrix}$ <br>
are adjacency matrix of *A b I b I* aph Product – Adjacency Matrix<br> **A** graph product has a predictably regular<br> **A** a diacency matrix of the product graph<br>  $A_3$  WB<sub>3</sub> =  $\begin{pmatrix} A & b_{1,2}I_3 & b_{1,3}I_3 \\ b_{2,1}I_3 & A & b_{2,3}I_3 \\ b_{3,1}I_3 & b_{3,2}I_3 & A \end{pmatrix}$ <br>
the a  $\det - \text{Adjacency Matrix} \ \text{ln} \ \text{but has a predictably regular} \ \text{not matrix of the product graph:} \ A \quad b_{1,2}I_3 \quad b_{1,3}I_3 \ b_{2,1}I_3 \quad A \quad b_{2,3}I_3 \ b_{3,1}I_3 \quad b_{3,2}I_3 \quad A \ \text{by matrix of the product graph is} \ \text{posed of one of the adjacency} \ \text{diagonal and identities of equal} \ \text{e other matrix' entries equal 1.}$ oduct – Adjacency Matrix<br>
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ncy matrix of the product graph is<br>
aposed of on

## Tridiagonal Block Matrices

- Because p-group adjacency matrices are always tridiagonal, (non-zero entries are restricted to the diagonal, superdiagonal, and subdiagonal) the adjacency matrix of arbitrarily many p-groups is always a tridiagonal block matrix.
- Thus, every cyclic group adjacency matrix has a tridiagonal block form.
- There is a formula to give the determinant of such matrices. [Molinari, 2008]

#### Prime Result

• Theorem: Given any cyclic group,  $\mathbf{Z}_n \cong \bigoplus \mathbf{Z}_{\mathbb{Z}_n}$ , its eigenvalues are  $\sum_{i=1}^{\infty} p_i^{\alpha_i}$ ,  $\sum_{i=1}^{\infty}$ *i s*  $n = \bigcup_{i=1}^{\infty} \mathbf{Z}_{p_i^{\alpha_i}}$ , its  $\boldsymbol{Z}_n \cong \bigoplus_{i=1}^s \boldsymbol{Z}_{p_i^{\alpha_i}}$ , its<br>=0

$$
\sum_{i=1}^s \{2\cos(\frac{j\pi}{\alpha_i+1})\}_{j=0}^{\alpha_i}
$$

- That is, its spectrum is the Cartesian sum of the spectra of all its component p-groups. Prime Result<br>
any cyclic group,  $\mathbf{Z}_n \approx \bigoplus_{i=1}^s \mathbf{Z}_i$ <br>
{ $2 \cos(\frac{j\pi}{\alpha_i+1})\}_{j=0}^{\alpha_i}$ <br>
cum is the Cartesian sum of component p-groups.<br>
atter for all cyclic groups.
- This settles the matter for all cyclic groups.

#### Elementary Abelian Groups

An *elementary Abelian group* is a group of the form

$$
(\mathbf{Z}_{p})^{n} = \bigoplus_{i=1}^{n} \mathbf{Z}_{p}
$$

• These are interesting because  $\mathbb{Z}_n$  is a field, which means that  $(\mathbf{Z}_p)^n$  is a finite vector space where subgroups correspond to subspaces. This allows us to more easily determine the subgroup lattice structure of these groups and also their spectra. entary Abelian  $grou$ <br>  $(Abelian group)$ <br>  $(\mathbf{Z}_p)^n = \bigoplus_{i=1}^n g(v_i)$ <br>  $(gv_i)$  is a finite victor in the summer the summer that **h**<br>**z**  $\bigcap_{p}$  *p*  $\bigcap_{i=1}^{n}$  *Z*  $\bigcap_{p}$  *<i>i*  $\bigcap_{i=1}^{n}$  **Z**  $\bigcap_{p}$ <br>**sting because <b>Z**<sub> $p$ </sub> is a field, which<br> $\bigcap_{p}$  is a finite vector space where<br>spond to subspaces. This allows us to<br>rmine the subgroup *p* (*Z*<sub>*p*</sub>)<sup>*n*</sup> =  $\bigoplus_{i=1}^{n}$  *Z*<sub>*p*</sub><br>teresting because *Z*<sub>*p*</sub> is a finite vector space correspond to subspaces. The determine the subgroup laterminate *p*

#### Elementary Abelian Groups (Cont'd)

- Elementary Abelian groups are also very interesting because they are the first case of the larger family of groups,  $(\mathbf{Z}_{n^n})^m$ , the carefully chosen subgroups of which, for distinct *p*, comprise all the coprime factors of any Abelian group. mentary Abelian Groups (Cont'd)<br>
ary Abelian groups are also very interesting<br>
they are the first case of the larger family of<br>  $(\mathbf{Z}_{p^s})^m$ , the carefully chosen subgroups of<br>
or distinct p, comprise all the coprime fa Elementary Abelian Groups (Cont'd)<br> **Elementary Abelian groups are also very interesting**<br>
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which, for dist *P* **Elementary Abelian Groups (Cont'd)**<br> **P Elementary Abelian groups are also very interesting because they are the first case of the larger family of groups,**  $(\mathbf{Z}_p,)^m$ **, the carefully chosen subgroups of which, for**  $\begin{array}{l} \displaystyle{\text{matrix A}}\\ \text{tary Abelian} \\ \text{they are the}\\ (\textbf{Z}_{p^n})^m, \text{the c:} \\ \text{or distinct } p, \text{belian} \end{array}$  $m<sub>th</sub>$   $\sim$   $\sim$   $\sim$  $p^n$ ,  $\sum$  $\mathbf{Z}_{n}$ <sup>n</sup>, the carefu
- Results:

\n- • Results:
\n- • 
$$
A_{(\mathbf{z}_p)^3} = (p - \lambda^2)^{(p^2 + p)} (2\lambda^4 - 7p^2\lambda^2 - 6p\lambda^2 - 3\lambda^2 + 4p^4 + 3p^3 - 4p^2 - 9p - 4)
$$
\n- • This gives the spectrum for all groups of the form  $(\mathbf{Z}_p)^3$
\n- • **This gives** the spectrum for all groups of the form  $(\mathbf{Z}_p)^3$
\n

- This gives the spectrum for all groups of the form **Z**
- These matrices are also tridiagonal in general, so this is promising.

## What's Next

- Continue exploring block matrix methods in solving all elementary Abelian groups.
- Expand upon these methods to solve all Abelian groups.
- Solve dihedral groups and other semidirect products.
- What can be said about non-isomorphic groups with cospectral subgroup lattices?
- Spectra of subring lattices?
- And for something completely different: what groups have subgroup lattices of genus 1 (can be drawn on a torus without intersection)?