Generalized Derangement Graphs

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- If *P* is a set, the bijection *f: P → P is a permutation* of *P*.

- Permutations can be written in *cycle notation* as the product of disjoint cycles:

  For example, (12)(34) is the permutation which sends 1→2, 2→1, 3→4, 4→3.

- The symmetric group S_n is the group of all possible permutations of *n* objects.

  Ex: S_3 = { e, (12), (13), (23), (123), 132)}

- If $\sigma \in S_n$, σ induces a permutation $\sigma_{(k)}$ on *k*-tuples by $\sigma_{(k)}(\{a_1, ..., a_k\}) = \{\sigma(a_1), ..., \sigma(a_k)\}$.

Ex: The permutation (1234) induces a permutation on *2*-tuples (pairs) as follows:

Permutation of 2-tuples

$$(1234)_{(2)}(\{1,2\}) = \{2,3\}$$

$$(1234)_{(2)}(\{1,3\}) = \{2,4\}$$

$$(1234)_{(2)}(\{1,4\}) = \{2,1\} = \{1,2\}$$

$$(1234)_{(2)}(\{2,3\}) = \{3,4\}$$

$$(1234)_{(2)}(\{2,4\}) = \{3,1\} = \{1,3\}$$

$$(1234)_{(2)}(\{3,4\}) = \{4,1\} = \{1,4\}]$$

- A permutation is an *ordinary derangement* (the set of which is denoted $\mathcal{D}_n$) if it has no fixed points.

  [$\mathcal{D}_n := \{\sigma \in S_n | \sigma(x) \neq x, \forall x \in [n]\}$]

- A permutation is a *k-derangement* (the set of which is denoted $\mathcal{D}_{k,n}$) if it leaves no *k*-tuple fixed.

- The number of *k*-derangements in S_n is denoted $D_k(n)$.

- Whether a permutation is a *k*-derangement or not depends only on its cycle structure.

  [For example, if the permutation (1234) is a *k*-derangement, then (1324), (1243), (2134), etc. will be as well.]

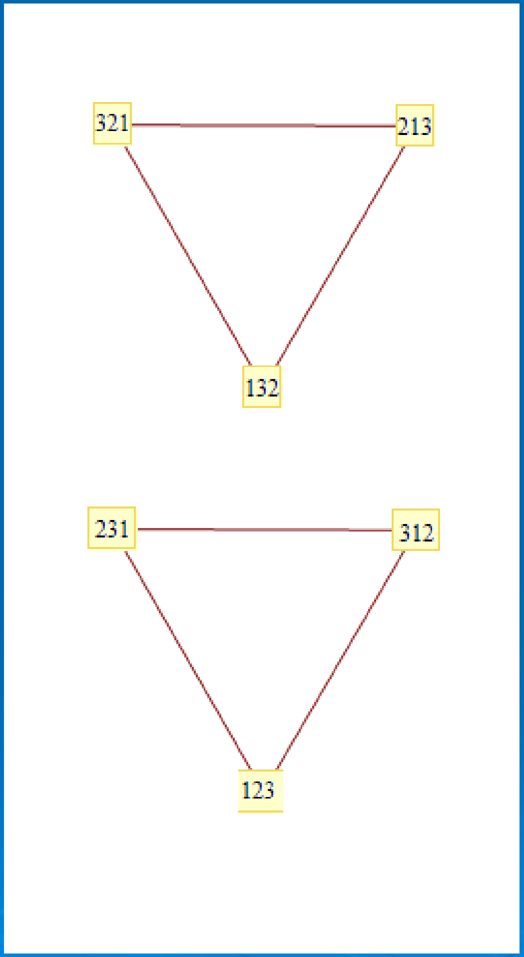
- In order for the permutations of a particular cycle structure to be *k*-derangements in S_n, the cycle structure must not partition *k.*

  Ex: (12)(34) is a *3*-derangement in S_4, but (12)(3)(4) is not, since {2,1} is a partition of 3.

Graphs!

- The *k-derangment graph* $\Gamma_{k,n}$ is the graph with the elements of S_n as its vertices, and an edge between two vertices iff they are *k*-derangements of one another.

1-derangement graph in S_3.



Properties of *k*-derangement graphs

- $\Gamma_{k,n}$ is $D_k(n)$-regular.
- $\Gamma_{k,n}$ is connected for $n > 3$.
- $\Gamma_{k,n}$ is Hamiltonian ($n > 3$).
- $\Gamma_{k,n}$ is Eulerian if and only if k is even or k and n are both odd. ($n > 3$).

Connected

- To show $\Gamma_{k,n}$ was connected we adapted a proof by Paul Renteln.
  - Every permutation is the product of adjacent transpositions (h,h+1)
  - Every adjacent transposition is the product of two derangements
- This means that the elements of $\mathcal{D}_{k,n}$ generate S_n, and so there's a path between the identity and every vertex of $\Gamma_{k,n}$. Thus $\Gamma_{k,n}$ is connected.

Eulerian

- Theorem: A graph *G* is Eulerian iff:
  - *G* is connected
  - Each vertex of *G* has even degree

- Lemma 1.1: If a permutation's cycle decomposition includes a cycle with length greater than 2, there are an even number of permutations with that cycle structure.

Hamiltonian

- To prove that $\Gamma_{k,n}$ is Hamilitonian, we utilized several existing theorems:
  - Jackson's Theorem: A 2-connected *h*-regular graph with no more than 3*h* vertices is Hamiltonian
  - Watkins' Theorem: If *G* is a connected, vertex transitive graph with vertex degree *d*, then the connectivity of *G* is at least 2*d*/3
- We still need to prove that $\Gamma_{k,n}$ has no more than $3D_k(n)$ vertices, but the numerical evidence shows that this is true.

Hamiltonian (cont.)

- It has been proven that ordinary derangement graphs have at most 3$D_1(n)$ vertices.

- To show that $\Gamma_{k,n}$ has no more than 3 $D_k(n)$ vertices, it is sufficient to show that

$$D_1(n) \leq D_k(n) \, \forall \, k, n$$

Hamiltonian (cont.)

- In order to show this, we are trying to find a 1-1 mapping from $D_1(n)$ to a subset of $D_k(n)$.
- There are some cycle structures which will both 1-derangements and *k*-derangements, so we map permutations with those cycle structures to themselves.
- So all we need to do is find a 1-1 mapping from the set of permutations which are 1-derangements but not *k*-derangements to a subset of the set of permutations which are 3-derangements but not 1-derangements.

THIS IS HARD.

Other Stuff

- Independence number = k!(n-k)!
  - We've got a proof written which gives us the lower bound for the independence number, but we need to know the clique number to get an upper bound.
- Clique number = ??
  - We believe that the clique number of $\Gamma_{k,n}$ will be $\binom{n}{k}$ for either an odd or prime *n*.
- Chromatic number = $\binom{n}{k}$
  - We found that the maximal independent set of $\Gamma_{k,n}$ containing the identity forms a group, and the rest of the maximal independent sets are its cosets.