Continuous Threshold Policy Harvesting in Predator-Prey Models

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Harvesting Model # 1

We begin with the predator-prey model given by:

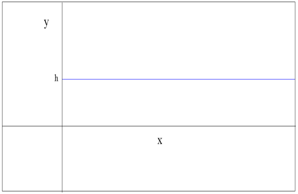
$$\begin{aligned} \dot{x} &= x(1-x) - \frac{axy}{1+mx} - H(x) \\ \dot{y} &= y\left(-d + \frac{bx}{1+mx}\right), \end{aligned} \tag{1}$$

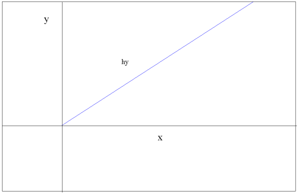
with a rational Threshold Policy Harvesting model:

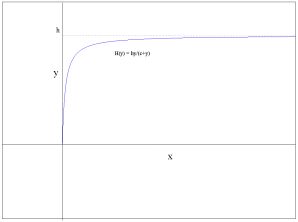
$$H(x) = \begin{cases} 0 & \text{if } x < T \\ \frac{h(x-T)}{h+x-T} & \text{if } x \geq T. \end{cases} \tag{2}$$

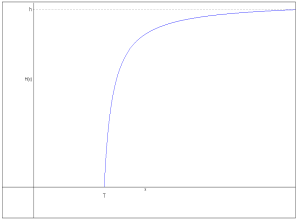
Here, $d =$ the death rate, $a =$ capture rate, $m =$ half saturation constant, $b =$ prey conversion rate, $h =$ harvesting constant, and $T =$ threshold constant.

Model Comparison









Nonlinear Systems

Nonlinear systems can be studied locally or globally.

Nonlinear Systems

Nonlinear systems can be studied locally or globally.
Local: Studying the system locally shows how the solutions behave around equilibria and periodic orbits.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \stackrel{linearization}{\Longrightarrow} \dot{\mathbf{x}} = A\mathbf{x}$$

where

$$A = J(\mathbf{x_0}) = \left. \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \cdots & \frac{\partial \dot{x}_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \dot{x}_n}{\partial x_1} & \cdots & \frac{\partial \dot{x}_n}{\partial x_n} \end{bmatrix} \right|_{\mathbf{x_0}}$$

Equilibrium Points when $x < T$

By solving the system of equations shown below, we computed the possible equilibrium points, or the values at which the system experiences stasis. Thus, solving

$$\begin{aligned} 0 &= x(1-x) - \tfrac{axy}{1+mx} - H(x) \\ 0 &= y\left(-d + \tfrac{bx}{1+mx}\right) \end{aligned}$$

when $x < T$, with $x, y > 0$, gives

$$P_1 = \left(\frac{d}{b-dm}, \frac{b}{a}\left[\frac{b-d(m+1)}{(b-dm)^2}\right]\right).$$

When $y = 0$,

$$P_2 = (1, 0).$$

Since x and y must be nonnegative, the parameters of each point must satisfy the condition that $b > d(m+1)$.

Equilibrium Points when $x \geq T$

In the case where $x \geq T$, when $x, y > 0$,

$$P_3 = \left(\frac{d}{b-dm}, \frac{hb(T - \frac{d}{b-dm})}{ad(h - T + \frac{d}{b-dm})} - \frac{b(d-b+dm)}{a(b-dm)^2} \right)$$

and when $y = 0$, the abscissa of P_4 is the solution of

$$x^3 + (h - T - 1)x^2 + Tx - Th = 0.$$

To determine the behavior and stability of the equilibrium points, we perform a trace-determinant analysis and an eigenvalue analysis on each point. For this, we call upon [1]:

Theorem 2 [Eigenvalue Analysis]

Let $\delta = \det A$ and $\tau = \text{trace } A$ and consider the linear system

$$\dot{x} = Ax$$

(a) If $\delta < 0$ there are two real eigenvalues of opposite sign. Then the system experiences a saddle.

(b) If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then there are two real eigenvalues of the same sign as τ and the equilibrium point is a node. It is stable if $\tau > 0$ and unstable if $\tau < 0$

(c) If $\delta > 0, \tau^2 - 4\delta < 0$, and $\tau \neq 0$ then there are two complex conjugate eigenvalues $\lambda = a \pm ib$ and the equilibrium point is a focus.

(d) If $\delta > 0$ and $\tau = 0$, then there are two purely imaginary complex conjugate eigenvalues and the equilibrium point is a center.

Likewise, the trace-determinant analysis we perform consists of similar statements.

Theorem 3 [Trace-Determinant Analysis]

Let $\delta = \det A$ and $\tau = \operatorname{trace} A$ and consider the linear system

$$\dot{x} = Ax$$

(I) If $\delta < 0$ then (1) has a saddle at the equilibrium point.

(II) If $\delta > 0$ and $\tau^2 - 4\delta \geq 0$ then (1) has a node at the equilibrium point; it is stable if $\tau < 0$ and unstable if $\tau > 0$.

(III) If $\delta > 0, \tau^2 - 4\delta < 0$, and $\tau \neq 0$ then (1) has a focus at the equilibrium point; it is stable if $\tau < 0$ and unstable if $\tau > 0$.

(IV) If $\delta > 0$ and $\tau = 0$, then (1) has a center at the equilibrium point.

Using the principles from Theorem 2, we found that P_1 can never be a saddle since $\delta_{P_1} < 0$ requires that $b < d(m+1)$. Likewise, if $b > d(m+1)$ and $\tau_{P_1}^2 - 4\delta_{P_1} \geq 0$, then P_1 is a node.

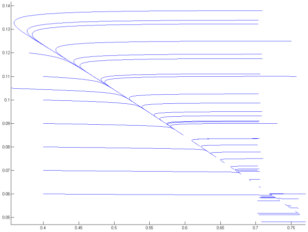


Figure: Node

If $b < \frac{dm(m+1)}{m-1}$, then the node is stable.

If $b > d(m+1)$ and $\tau_{P_1}^2 - 4\delta_{P_1} < 0$, then P_1 is a focus. As before, if $b < \frac{dm(m+1)}{m-1}$, then the focus is stable.

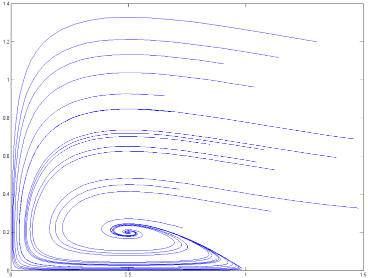


Figure: Focus

Finally, P_1 is a center if $b = \frac{md(m+1)}{m-1}$.

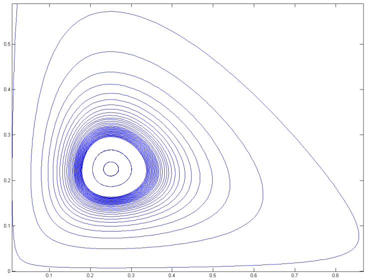


Figure: Center

Hopf Bifurcation

When P_3 satisfies the conditions to be a center, the system exhibits a Hopf Bifurcation.

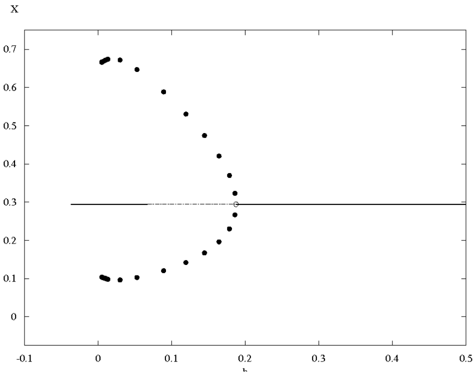
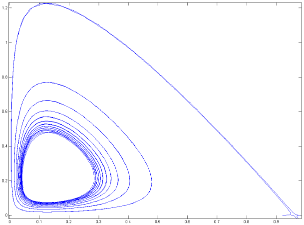


Figure: Bifurcation Diagram

Heteroclinic Orbit

Additionally, we detected a heteroclinic orbit going from P_4 as a saddle point toward the periodic orbits of the Hopf bifurcation.



Harvesting Model # 2

We use the same system of equations as before,

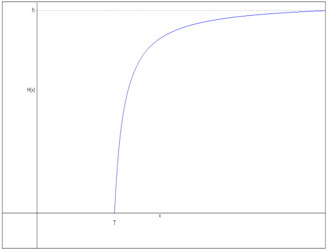
$$\begin{aligned} \dot{x} &= x(1-x) - \frac{axy}{1+mx} - H(x) \\ \dot{y} &= y\left(-d + \frac{bx}{1+mx}\right), \end{aligned} \tag{3}$$

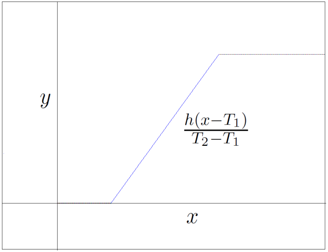
with a slightly different Threshold Policy Harvesting model:

$$H(x) = \begin{cases} 0 & \text{if } x < T_1 \\ \frac{h(x-T_1)}{T_2-T_1} & \text{if } T_1 \leq x < T_2 \\ h & \text{if } x \geq T_2. \end{cases} \tag{4}$$

As before, $d =$ the death rate, $a =$ capture rate, $m =$ half saturation constant, $b =$ prey conversion rate, $h =$ harvesting constant, $T_1 =$ the initial threshold constant, and $T_2 =$ the final threshold constant.

Rational vs. Piecewise Linear





Equilibrium Points when $T_1 \leq x \leq T_2$

By solving the same system of equations as in the first model, we computed the possible equilibrium points of the second model. Solving this, when $T_1 \leq x \leq T_2$, with $x, y > 0$, gives

$$P_5 = \left(\frac{d}{b - dm}, \frac{b}{a} \left[\frac{b - dm - d}{(b - dm)^2} - \frac{h(d - T_1(b - dm))}{d(b - dm)(T_2 - T_1)} \right] \right).$$

When $y = 0$,

$$P_6 = (x_+^*, 0) \quad \text{and} \quad P_7 = (x_-^*, 0),$$

where $x_\pm^* = \frac{T_2 - T_1 - h \pm \sqrt{(h - T_2 + T_1)^2 + 4hT_1(T_2 - T_1)}}{2(T_2 - T_1)}$.

Since x and y must be nonnegative, the parameters of each point must satisfy the conditions that

$$b > dm \quad \text{and} \quad h < \frac{d(b - dm - d)(b - dm)(T_2 - T_1)}{d - T_1(b - dm)}.$$

Equilibrium Points when $x \geq T_2$

In the case where $x \geq T_2$, when $x, y > 0$,

$$P_8 = \left(\frac{d}{b - dm}, \frac{b}{a} \left[\frac{b - dm - d}{(b - dm)^2} - \frac{h}{d} \right] \right),$$

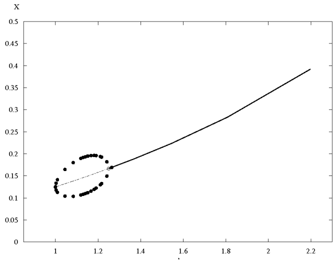
and when $y = 0$,

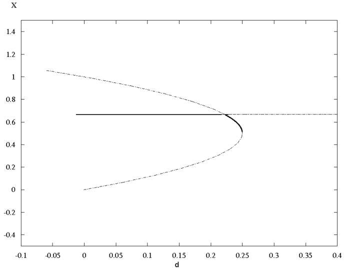
$$P_9 = (\hat{x}_+, 0) \quad \text{and} \quad P_{10} = (\hat{x}_-, 0),$$

where $\hat{x}_\pm = \frac{-1 \pm \sqrt{1-4h}}{-2}$.

Bifurcations and Heteroclinic Orbits

- When P_5 satisfies the conditions to be a center, then the system exhibits a Hopf Bifurcation when d=1.25.
- In P_8, P_9, and P_{10}, both a fold bifurcation and a transcritical bifurcation were detected at h=0.25 and h=0.22, respectively.
- Heteroclinic orbits are present, connecting equilibrium points to the periodic orbits coming out from Hopf bifurcations.





Diffusive Predator-Prey Model

We took a model similar to that of system (1), (4) and added a diffusion term to the predator species to get a new model:

$$\frac{du}{dt} = u(1-u) - \frac{auv}{1+mu} - H(u)$$
$$\frac{dv}{dt} = v(-d + \frac{bu}{1+mu}) + V_{xx}.$$

Using a change of variables, we made this system into an ODE:

$$\begin{aligned} \dot{u} &= \frac{du}{dz} = \frac{1}{s}\left(u(1-u) - \frac{auv}{1+mu} - H(u)\right) \\ \dot{v} &= \frac{dv}{dz} = w \\ \dot{w} &= \frac{dw}{dz} = sw + v(d - \frac{bu}{1+mu}), \end{aligned} \qquad (5)$$

Discussion

- We seek to illustrate how the different harvesting functions affect the qualitative behaior of the solutions and the stability of the equilibria by:
  - Analyzing the equilibrium points and their properties for both models
  - When $x, y > 0$.
- In such a case,
  - We always have $x = \frac{d}{b-dm}$
  - The corresponding y values satisfy:

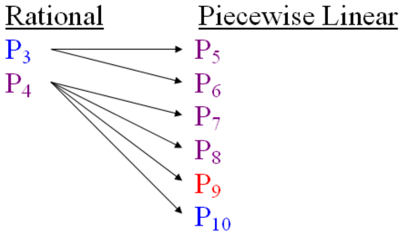
$$y_1 \geq y_3 \geq y_5 \geq y_8 \tag{6}$$

when

$$\frac{T(b-dm)-d}{(b-dm)(h-T)+d} \geq \frac{T_1(b-dm)-d}{(b-dm)(T_2-T_1)}.$$

Discussion

- We found two main types of behavior in the group of equilibria:
  - All equilibria on the x-axis can be nodes or saddles but never centers or foci
  - All other equilibria can be nodes, foci, and centers, but not saddles.
- The equilibria of the second model correspond directly to the equilibria of the first model:



Possible Extensions

- Harvesting on both predators and prey
- Combining threshold policy harvesting with other types of harvesting (i.e. seasonal harvesting)
- Including an infectious disease, especially concerning a risk of mortality of one or both species
- Considering more than two species

Questions

Are there any questions?

References

L. Perko, Differential Equations and Dynamical Systems, Texts in Appl. Math. Vol 7, Springer Verlag (2006).