Generalized Derangements

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Overview

- In this presentation we will address the following:
  - Introductory definitions
  - Classic derangements
  - *k*-derangements
  - Exponential generating functions
  - Limits
  - Rencontres numbers
  - k-rencontres numbers

Introductory Definitions

- Permutation – A mapping of elements from a set to elements of the same set. Ex. $\sigma : \{1,2,3,4,5\} \rightarrow \{2,5,4,3,1\}$

- Cycle – A permutation of the elements of some set *X* which maps the elements of some subset *S* to each other in a cyclic fashion, while fixing all other elements.

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 5 & 4 & 3 & 1 \end{matrix} = (125)(34)$$

- Partition – A set of nonempty subsets of *X* such that every element *x* in *X* is in exactly one of these subsets.

  - Ex. $\{\{1,2,3\},\{1,2\}\{3\},\{1\}\{2\}\{3\}\}$

What Are Classic Derangements?

- The classic derangement problem asks, "How many permutations of *n* objects leave no elements fixed?"
  - For example, "If six students in a class all take a test, how many ways can the teacher pass back their exams so that none of the students receive their own test?"
- The solution to this problem can be found by using one of the well known classic derangement formulas.

D_n satisfies the following recurrence:

$$D_0 = 1 \text{ and } D_1 = 0,$$
$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

Or explicitly

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \quad (1)$$

- Using (1) it is clear to see that $D_6 = 265$.
- From here we can compute the probability that a permutation is a derangement by dividing by the total number of possible permutations, *n*! Therefore, the probability that no student receives their own test in our example is $D_6/6! \approx .368$
- It is also well known that: $\lim_{n\to\infty} \frac{D_n}{n!} = \frac{1}{e} \approx .3679$

What Are *k*-Derangements?

- Definition:

Let $[n] = \{1, 2, 3, \ldots, n\}$ and S_n be the symmetric group of all permutations of [n]. Let σ be a permutation of [n] within S_n. Then, let P_r be the set of all permutations in S_n whose cycle structure is r. [Ex. $P_{\{2,1\}} = \{(12), (13), (23)\}$] Also, let $A^{(k)}$ be the set of unordered k-tuples with elements from A. [Ex. $A = \{1, 2, 3\}$, $A^{(2)} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$] Therefore, since a permutation of the elements of [n] induces a permutation of unordered k-tuples, a permutation $\sigma_{(k)} \in S_n$ is a k-derangement $D_{(k,n)}$ if $\{\sigma_{(k)}(x) \neq x \,|\, \forall\, x \in [n]^{(k)}\}$

- In other words, a permutation is a *k* -derangement if its cycle decomposition does not have any cycles whose lengths partition *k*.
  - For example, $P_{\{2,2\}}$ is a 3-derangement in S_4 since neither of the cycles of length 2 can partition *k*=3. However, $P_{\{2,1,1\}}$ is not a 3-derangement since either combination of the cycles of length {2,1} partition *k*=3.
- Being able to calculate *k*-derangements will allow us to ask questions such as, "How many ways can a teacher pass back a test such that no two students receive their own test or each others tests?"

Calculating *k*-Derangements

- To calculate the number of *k*-derangements by hand:
  - Start with the number of all permutations - $n!$
  - Divide by the length of each partition.
  - Distinguish between repeated partitions by dividing by $r!$ where r is the number of times each partition is repeated.
  - Repeat this process and sum over all acceptable permutations of P_r.
- Ex. $D_{(2,6)}$ $\quad P_{\{6\}}$ $\quad P_{\{5,1\}}$ $\quad P_{\{3,3\}}$

$$\frac{6!}{6} = 120 \qquad \frac{6!}{5 \cdot 1} = 144 \qquad \frac{6!}{3 \cdot 3 \cdot 2!} = 40$$

$$D_{(2,6)} = 120 + 144 + 40 = 304$$

Table of *k*-Derangements

$D_{(k,n)}$	$D_{(1,n)}$	$D_{(2,n)}$	$D_{(3,n)}$	$D_{(4,n)}$	$D_{(5,n)}$
$D_{(k,1)}$	0	1	1	1	1
$D_{(k,2)}$	1	0	2	2	2
$D_{(k,3)}$	2	2	0	6	6
$D_{(k,4)}$	9	14	9	0	24
$D_{(k,5)}$	44	54	54	44	0
$D_{(k,6)}$	265	304	459	304	265
$D_{(k,7)}$	1854	2260	2568	2568	2260
$D_{(k,8)}$	14833	18108	20145	26704	20145

$$D_{(k,n)} = D_{(n-k,n)}$$

Exponential Generating Functions

- A generating function is a formal power series whose coefficients encode information about a sequence a_n.
- Exponential generating functions are represented as:

$$EG(a_n; x) = \sum_{n=o}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} \dots$$

- **Proposition:** *Let C_l represent cycles of length l. Then the exponential generating function for C_l is the power series, centered at 0, whose coefficients of x^n represent n numbers of C_l.*

$$EG(C_l; x) = (1 + x^l/l + \frac{(x^l/l)^2}{2!} + \frac{(x^l/l)^3}{3!} + \dots) = e^{(\frac{x^l}{l})}$$

$$EG(C_1; x) = e^x \qquad EG(C_2; x) = e^{(x^2/2)} \qquad EG(C_3; x) = e^{(x^3/3)}$$

$EG(P;x)$ and $EG(D_{(1,n)};x)$

- Since permutations can be represented as a product of disjoint cycles, the exponential generating function for all permutations is simply the multiplication of the EG's for all of the cycles.

$$EG(P;x) = e^x e^{(x^2/2)} x^{(x^3/3)} \ldots = \frac{1}{1-x}$$

- Also, since classic derangements cannot contain permutations whose cycle decompositions contain cycles of length 1, we can remove the EG for 1-cycles to obtain the EG for classic derangements.

$$EG(D_{(1,n)};x) = e^{(x^2/2)} x^{(x^3/3)} \ldots = \frac{e^{-x}}{1-x}$$

EG's for k-Derangements

- Just as we found the EG for classic derangements, we can find the EG's for *k*-derangements by removing from the EG of all permutations those cycles whose lengths partition *k*.

$$EG(D_{(1,n)};x) = \frac{e^{-x}}{1-x}$$

$$EG(D_{(2,n)};x) = \frac{1}{1-x}e^{-x-\frac{x^2}{2}}(1+x)$$

$$EG(D_{(3,n)};x) = \frac{1}{1-x}e^{-x-\frac{x^2}{2}-\frac{x^3}{3}}(e^{\frac{x^2}{2}}+x+\frac{x^2}{2!})$$

$$EG(D_{(4,n)};x) = \frac{1}{1-x}e^{-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}}(e^{\frac{x^3}{3}}(1+\frac{x^2}{2})+x(1+\frac{x^2}{2})+\frac{x^2}{2!}+\frac{x^3}{3!})$$

- We have found the EG's for k-derangements up to k=8, however we have not yet been able to generalize a pattern from the EG's since they become increasingly complex.

Formulas for 2-Derangements

- By performing the series expansion on

$$EG(D_{(2,n)};x) = \frac{1}{1-x}e^{-x-\frac{x^2}{2}}(1+x)$$

it can be shown that:

$$D_{(2,n)} = n!\sum_{i=0}^{n}\left(\sum_{k=\lceil\frac{n}{2}\rceil}^{n}\frac{(-1)^k}{(n-k)!(2k-n)!2^{(n-k)}} + \sum_{k=\lceil\frac{n-1}{2}\rceil}^{n-1}\frac{(-1)^k}{((n-1)-k)!(2k-(n-1))!2^{((n-1)-k)}}\right)$$

- $D_{(2,n)}$ also satisfies the following recurrence:

$$D_{(2,0)} = 1 \quad D_{(2,1)} = 1 \quad D_{(2,2)} = 0 \quad D_{(2,3)} = 2$$

$$D_{(2,n)} = D_{(2,n-1)} + (n-1)(n-3)D_{(2,n-2)} + (n-1)(n-2)D_{(2,n-3)} + (n-1)(n-2)(n-3)D_{(2,n-4)}$$

Limits of EG's

- Lemma

$$\text{If } \frac{f(x)}{(1-x)} = \sum_{n=0}^{\infty} q_n x^n \text{ , then } \lim_{n\to\infty} q_n = f(1).$$

- Proof

Let $f(x) = (b_0 + b_1 x + b_2 x^2 \dots)$ then it follows that:

$$f(x)\frac{1}{1-x} = (b_0 + b_1 x + b_2 x^2 \dots)(1 + x + x^2 + x^3 + \dots)$$

$$= b_0 + (b_0 + b_1)x + (b_0 + b_1 + b_2)x^2 \dots$$

therefore

$$q_n = (b_0 + b_1 + b_2 + \cdots + b_n)$$

$$\lim_{n\to\infty} q_n = \sum_{i=0}^{\infty} b_i = f(1)$$

Limits of $D_{(k,n)}/n!$

- Using the previous lemma, we can find the limits of the first few values of $D_{(k,n)}/n!$ as n goes to infinity.

$$\lim_{n\to\infty} \frac{D_{(1,n)}}{n!} = \frac{1}{e} \approx .367879$$

$$\lim_{n\to\infty} \frac{D_{(2,n)}}{n!} = \frac{2}{e^{3/2}} \approx .44626$$

$$\lim_{n\to\infty} \frac{D_{(3,n)}}{n!} = \frac{3}{2e^{11/6}} + \frac{1}{e^{4/3}} \approx .503417$$

$$\lim_{n\to\infty} \frac{D_{(4,n)}}{n!} = \frac{13}{6e^{25/12}} + \frac{3}{2e^{7/4}} \approx .530442$$

$$\lim_{n\to\infty} \frac{D_{(5,n)}}{n!} \approx .558525$$

$$\lim_{n\to\infty} \frac{D_{(6,n)}}{n!} \approx .574941$$

$$\lim_{n\to\infty} \frac{D_{(7,n)}}{n!} \approx .591519$$

$$\lim_{n\to\infty} \frac{D_{(8,n)}}{n!} \approx .602722$$

- Conjecture – $\lim_{k\to\infty} \lim_{n\to\infty} \frac{D_{(k,n)}}{n!} = 1$

Rencontres Numbers

- Definition - For $n \geq 0$ and $0 \leq r \leq n$, the rencontres number $D_{(r,n)}$ is the number of permutations of $[n]$ that have exactly r fixed points.

- In other words, rencontres numbers specify how many permutations leave r number of 1-tuples fixed.

- Note $D_{(0,n)}$ are classic derangements.

- Known formulas for rencontres numbers include:

  - $D_{(r,n)} = \binom{n}{r} \cdot D_{(0,n-k)}$

  - $EG(D_{(r,n)}; x) = \frac{x^r}{r!} \frac{e^{-x}}{1-x}$

k-Rencontres Numbers

- Definition - For $n \geq 0$, $0 \leq r \leq n/k$, and , $0 \leq k \leq n$ *k*-rencontres numbers $D_{(r,k,n)}$ are the number of permutations that leave exactly r number of *k*-tuples fixed.

- Note $D_{(0,k,n)}$ are *k*-derangements.

- We have found EG's for the following:

$$EG(D_{(r,1,n)};x) = \frac{x^r}{r!}\frac{e^{-x}}{1-x}$$

$$EG(D_{(r,2,n)};x) = \frac{1}{1-x}e^{-x-\frac{x^2}{2}}\sum_{i=0}^{\infty}\frac{(x^2/2)^{r-\binom{i}{2}}}{(r-\binom{i}{2})!}\frac{x^i}{i!}$$

$$EG(D_{(r,3,n)};x) =$$

$$\frac{1}{1-x}e^{-x-\frac{x^2}{2}-\frac{x^3}{3}}\left(\frac{(x^3/3)^r}{r!}e^{(x^2/2)}+\sum_{i=1}^{r+2}\sum_{j=0}^{\left\lceil\frac{r-\binom{i}{3}}{i}\right\rceil}\frac{(x^3/3)^{r-(ji+\binom{i}{3})}}{(r-(ji+\binom{i}{3}))!}\frac{(x^2/2)^j}{j!}\frac{x^i}{i!}\right)$$

Unsolved Problems

- Prove $\lim_{k\to\infty} \lim_{n\to\infty} \frac{D_{(k,n)}}{n!} = 1$
- Prove $D_{(k,n)} \equiv 0 \mod k$
- Find a recursive formula, explicit formula, or exponential generating function for all *k*-derangements.
- Find a recursive formula, explicit formula, or exponential generating function for all *k*-rencontres numbers.