Existence of a Limit on a Dense Set, and Construction of Continuous Functions on Special Sets

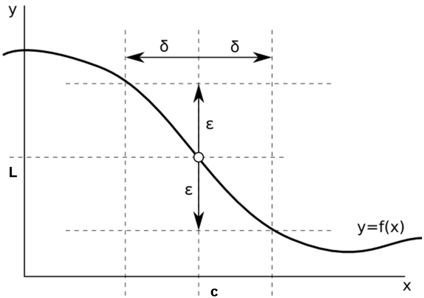
Goodi Khalsa, Charles Ouyang, and Hannah Skavdal

REU 2012

Recap: Definitions

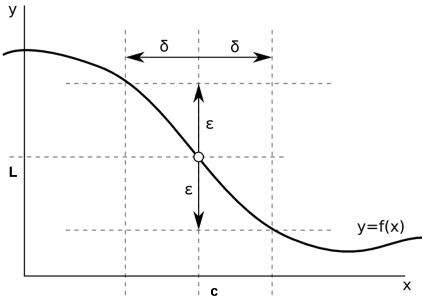
Definition

Given a real-valued function f, the limit of f exists at a point $c \in \mathbb{R}$ if for each given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in \mathbb{R}$ if $0 < |c - x| < \delta$, then $|f(c) - L| < \varepsilon$.



Definition

Given a real-valued function f, f is continuous at a point $c \in \mathbb{R}$ if for each given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in \mathbb{R}$ if $|c - x| < \delta$, then $|f(c) - f(x)| < \varepsilon$.



Definition

Let B be a subset of $\mathbb{R}$. B is said to be dense in $\mathbb{R}$ if for any point $x \in \mathbb{R}$, x is either in B or a limit point of B. Equivalently, given any $\varepsilon > 0$, and any $x \in \mathbb{R}$, then $\exists p \in B$ such that $p \in N_\varepsilon(x)$.

Definition

Let B be a subset of $\mathbb{R}$. B is said to be dense in $\mathbb{R}$ if for any point $x \in \mathbb{R}$, x is either in B or a limit point of B. Equivalently, given any $\varepsilon > 0$, and any $x \in \mathbb{R}$, then $\exists p \in B$ such that $p \in N_\varepsilon(x)$.

Notation

Given $f : [a, b] \to \mathbb{R}$, let F_C denote the set of points where f is continuous. Let F_+ denote the set of points where the right-sided limit of f exists. Similarly, let F_- denote the set of points where the left-sided limit of f exists. Then, also let F_L designate the set of points where a one-sided limit exists, that is $F_L = F_+ \cup F_-$.

Notation

Let D be the set of points of discontinuity of a real function f. Then each point in D belongs to one of the following 3 sets:

- D_1, *the set of points* $c \in [a, b]$ *such that* f *has either a removable or jump discontinuity at* c.
- D_2, *the set of points* $c \in [a, b]$ *such that* f *has only either a right-sided limit or a left-sided limit at* c.
- D_3, *the set of points* $c \in [a, b]$ *such that* f *has neither a right-sided limit or a left-sided limit at* c.

Definition

A set A is countable if there exists a one-to-one correspondence from A to $\mathbb{N}$.

Recap: Results

Proposition

Let f be a function defined on $[a, b]$. Then the sets D_1 and D_2 are at most countable.

Theorem

Let f be a function defined on $[a, b]$. Assume that the set F_L is dense in $[a, b]$. Then the set F_C is nonempty, dense in $[a, b]$, and uncountable.

Theorem

Let $f : [a, b] \to \mathbb{R}$, with F_L dense in $\mathbb{R}$. Then f is unique on F_C.

Corollary

If $f : [a, b] \to \mathbb{R}$ *and* $m(D_3) = 0$, *and given* F_L *dense in* $\mathbb{R}$, *then* f *is unique except at a set of measure zero.*

Proposition

Given a function $f : [a, b] \to \mathbb{R}$, there is no countable dense set G where $G := \{c : \lim_{x \to c} f(x) = L \text{ with } L \in \mathbb{R}\}$.

Definition

An F_σ set is a countable union of closed sets.
A G_δ set is a countable intersection of open sets.
Examples:
$\mathbb{Q}$ is F_σ.
$\mathbb{R} \backslash \mathbb{Q}$ is G_δ.

Definition

An F_σ set is a countable union of closed sets.
A G_δ set is a countable intersection of open sets.
Examples:
$\mathbb{Q}$ is F_σ.
$\mathbb{R}\backslash\mathbb{Q}$ is G_δ.

The set of continuity points of a function $f : \mathbb{R} \to \mathbb{R}$ is a G_δ set. Conversely, every G_δ subset of $\mathbb{R}$ is the set of continuity points of a function $f : \mathbb{R} \to \mathbb{R}$.

Theorem

$(F_L)^c$ is an F_σ set.

Theorem

$(F_L)^c$ is an F_σ set.

$$\text{Let } \omega(f;J) = \sup_{x\in J} f(x) - \inf_{x\in J} f(x)$$

Theorem

$(F_L)^c$ is an F_σ set.

$$\text{Let } \omega(f;J) = \sup_{x\in J} f(x) - \inf_{x\in J} f(x)$$

$$\omega(f;a) = \inf_{a\in J}(\omega(f;J))$$

Theorem

$(F_L)^c$ is an F_σ set.

$$\text{Let } \omega(f; J) = \sup_{x \in J} f(x) - \inf_{x \in J} f(x)$$

$$\omega(f; a) = \inf_{a \in J} (\omega(f; J))$$

$$E_r = \{a : \omega(f; a) \geq \tfrac{1}{r}\}$$

Theorem

There is no function that is continuous only on $\mathbb{Q}$.

Is there a G_δ set of measure zero, containing $\mathbb{Q}$, that we can construct a continuous function on?

Definition

A real number x is a Liouville number if for any given $n \in \mathbb{N}$, there exist infinitely many relatively prime integers p and q with $q > 1$ such that $0 < |x - \frac{p}{q}| < \frac{1}{q^n}$. We will denote the set of Liouville numbers with L.

Is there a G_δ set of measure zero, containing $\mathbb{Q}$, that we can construct a continuous function on?

Is there a G_δ set of measure zero, containing $\mathbb{Q}$, that we can construct a continuous function on?

Let $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0, & \textit{if } x \in L \cup \mathbb{Q} \\ \frac{1}{N_x}, & \textit{otherwise} \end{cases}$$

Where N_x is defined as the first n where x fails to meet the definition of a Liouville number. In other words, the first n such that there do not exist p and q such that $0 < |x - \frac{p}{q}| < \frac{1}{q^n}$.

Definition

The irrationality measure for a real number x is a numeric representation of how well x can be approximated by the rationals. Let μ be the least upper bound such that $0 < |x - \frac{p}{q}| < \frac{1}{q^{\mu}}$ where $p, q \in \mathbb{Z}$. We call μ the irrationality measure of x.

Notation

Let $\mu(x)$ stand for irrationality measure of x.

For example:
$\mu(x) = \infty$ if $x \in L$,
$\mu(x) = 1$ if $x \in \mathbb{Q}$,
$\mu(x) = 2$ if x is an algebraic number of degree > 1,
$\mu(x) \geq 2$ if x is transcendental.

For example:
$\mu(x) = \infty$ if $x \in L$,
$\mu(x) = 1$ if $x \in \mathbb{Q}$,
$\mu(x) = 2$ if x is an algebraic number of degree > 1,
$\mu(x) \geq 2$ if x is transcendental.

Definition

If r is a root of a nonzero polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_i \in \mathbb{Z}$ and r satisfies no similar equation of degree $< n$ then r is said to be an algebraic number of degree n. A number that is not algebraic is said to be transcendental.

Notation

For fixed n, $S_{n,k} = \left\{ x : |x - r_k| < \frac{1}{q_k^n} \right\}$ *where* r_k *is the rational* $\frac{p_k}{q_k}$. $C = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} S_{n,k}$.

Notation

For fixed n, $S_{n,k} = \left\{ x : |x - r_k| < \frac{1}{q_k^n} \right\}$ *where* r_k *is the rational* $\frac{p_k}{q_k}$. $C = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} S_{n,k}$.

$$B_N = \bigcup_{k=N}^{\infty} S_{n,k}, \quad B_1 \supset B_2 \supset \cdots \supset B_N \supset B_{N+1} \supset \cdots.$$

Notation

For fixed n, $S_{n,k} = \left\{ x : |x - r_k| < \frac{1}{q_k^n} \right\}$ *where* r_k *is the rational* $\frac{p_k}{q_k}$. $C = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} S_{n,k}$.

$$B_N = \bigcup_{k=N}^{\infty} S_{n,k}, \quad B_1 \supset B_2 \supset \cdots \supset B_N \supset B_{N+1} \supset \cdots.$$

$B = \bigcup_{k=1}^{\infty} S_{n,k}$. *We will be working with* B^c, *and* $B_N \backslash B_{N+1}$.

Notation

For fixed n, $S_{n,k} = \left\{ x : |x - r_k| < \frac{1}{q_k^n} \right\}$ *where* r_k *is the rational* $\frac{p_k}{q_k}$. $C = \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} S_{n,k}$.

$$B_N = \bigcup_{k=N}^{\infty} S_{n,k}, \quad B_1 \supset B_2 \supset \cdots \supset B_N \supset B_{N+1} \supset \cdots.$$

$B = \bigcup_{k=1}^{\infty} S_{n,k}$. *We will be working with* B^c, *and* $B_N \backslash B_{N+1}$.

$$\mathbb{R} = B^c \cup (B_1 \backslash B_2) \cup (B_2 \backslash B_3) \cup \cdots \cup C.$$

Proposition

For fixed n, define

$$h_n(x) = \begin{cases} 0, & \text{if } x \in C \cup B^c \\ \frac{1}{N}, & \text{if } x \in B_N \backslash B_{N+1} \end{cases}$$

$h_n(x)$ *is continuous on* $C \cup B^c$.

Proposition

For fixed n, $h_n(x)$ *is discontinuous on* $B_N \backslash B_{n+1}$ *for all* N.

Consider the function h.

$$h(x) = \lim_{n \to \infty} h_n(x)$$

Consider the function h.

$$h(x) = \lim_{n\to\infty} h_n(x)$$

$$g_n(x) = \begin{cases} 0, & \mathit{if}\ x \in C \cup B^c \\ \frac{1}{q_N}, & \mathit{if}\ x \in B_N \backslash B_{N+1} \end{cases}$$

then

$$\lim_{n\to\infty} g_n(x) = R(x)$$

Example

Dirichlet Function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Example

Riemann Function

$$R(x) = \begin{cases} \frac{1}{q}, & x \in \mathbb{Q}, x = \frac{p}{q} \text{ in lowest terms} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Theorem

(Egorov's Theorem) Let (X, B, μ) *be a measurable space and let* E *be a measurable set with* $\mu(E) < \infty$. *Let* f_n *be a sequence of measurable functions on* E *such that each* f_n *is finite almost everywhere in* E *and* f_n *converges almost everywhere in* E *to a finite limit. Then for every* $\varepsilon > 0$, *there exists a subset* A *of* E *with* $\mu(E - A) < \varepsilon$ *such that* f_n *converges uniformly on* A.