Dynamics and Bifurcations in Predator-Prey Models with Refuge, Dispersal and Threshold Harvesting

Alexander Hare and Keilah Ebanks

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Last Model

$$\begin{aligned} \dot{x} &= \alpha x(1-x) - \frac{a(1-m)xy}{1+c(1-m)x} - H(x) \\ \dot{y} &= -dy + \frac{b(1-m)xy}{1+c(1-m)x} \end{aligned} \quad (1)$$

where $H(x) = h$

each of α, a, b, c, d, h, m and b are positive real parameters

Continuation

New $H(x)$

$$H(x) = \begin{cases} 0 & x < T_1 \\ \frac{h(x-T_1)}{T_2-T_1} & T_1 \leq x \leq T_2 \\ h & x > T_2 \end{cases} \tag{2}$$

3-D Model:

$$\begin{aligned} \dot{x} &= \alpha x(1-x) - \frac{a(1-m)xy}{1+c(1-m)x} + D_1(v-x) \\ \dot{v} &= -d_v v + D_2(x-v) \\ \dot{y} &= -dy + \frac{b(1-m)x_1 y}{1+c(1-m)x_1 y} \end{aligned} \tag{3}$$

Saddle Point

Given a dynamical system $\dot{x} = Bx$, specifically at the solution $x(t) = e^{Bt}x_0$ if the eigenvalues of B are real with opposite sign, the point x_0 is a saddle point.

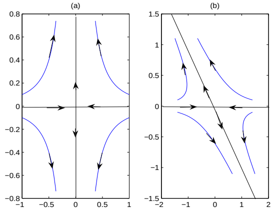

Figure: Example of Saddle Point

Node

If the eigenvalues are reals that have the same sign, the point is a node.

If both are positive, then the point is unstable.

If negative, (asymptotically) stable.

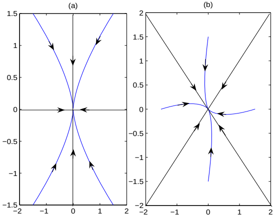

Figure: Example of Node

Focus

If the eigenvalues are complex conjugates, the point is a focus.
If the real parts are positive, the point is unstable.
If the real parts are negative, the point is (asymptotically) stable.

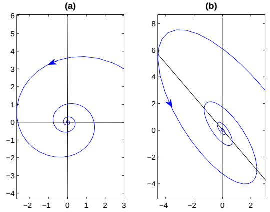

Figure: Example of Focus

Center

If the eigenvalues are purely imaginary, then the equilibrium point is of center type. This also indicates that the equilibrium is non-hyperbolic.

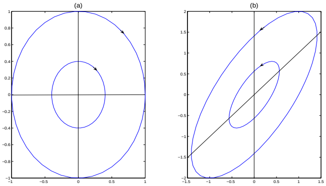

Figure: Example of Center

Non-Hyperbolic Points

This method only works for hyperbolic equilbrium points (from Hartman-Grobman Theorem)

For non-hyperbolic equilibrium points and some global analysis we need to perform bifurcation analysis.

Hartman-Grobman:
Indicates that near a hyperbolic equilibrium point x_0, the nonlinear system $\dot{x} = f(x)$ has the same qualitative structure as the linear system $\dot{x} = Ax$ with $A = Df(x_0)$

Model 1: $H(x) = h$

Analyze the model by finding equilibrium points and their stability.
3 equilibrium points:

$$P_0 = \left(\frac{\alpha - \sqrt{-4\alpha h + \alpha^2}}{2\alpha},\ 0 \right), \quad P_1 = \left(\frac{\alpha + \sqrt{-4\alpha h + \alpha^2}}{2\alpha},\ 0 \right),$$

$$P_2 = \left(\frac{d}{(b - cd)(1 - m)},\ b\frac{-\frac{h}{d} - \left(\frac{b(m-1)+d(1+c-cm)\alpha}{(b-cd)^2(m-1)^2}\right)}{a} \right)$$

2 boundary points on the x-axis (predator extinction)
1 interior boundary point (coexistence of the species)

Trace-Determinant Analysis

The Jacobian $J(x, y)$ is

$$\begin{bmatrix} \frac{acxy(m-1)^2}{z^2} - \alpha(x-1) - \frac{ay(m-1)}{z} - \alpha x & -\frac{ax(m-1)}{z} \\ \frac{by(m-1)}{z^2} & \frac{bx(m-1)}{z} - d \end{bmatrix}$$

where $z = cx(m-1) - 1$.

Conditions

if $D < 0$, point is a saddle
if $D > 0$
  $T^2 - 4D \geq 0$, point is a node
    $T > 0$, unstable
    $T < 0$, stable
  $T^2 - 4D < 0$, point is a focus
    $T > 0$, unstable
    $T < 0$, stable
  $T = 0$, point is of center-type (non-hyperbolic)

We want to look for conditions on our parameters that determine which type of equilibrium point is present.

Boundary Points

The Jacobians evaluated at P_0 and P_1 are (respectively)

$$\begin{bmatrix} \alpha - \alpha^3 + \alpha^2(\sqrt{\alpha^2 - 4\alpha h}) & -\frac{a}{c} - \frac{a}{c\left(\alpha c(\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-4\alpha h}}{2}(m-1))\right)} \\ 0 & \frac{\alpha b\left[\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\alpha h}}{2}\right](m-1)}{\alpha c[\frac{\alpha}{2} + \frac{\sqrt{\alpha^2-4\alpha h}}{2}](m-1)-1} - d \end{bmatrix}$$

$$\begin{bmatrix} (\alpha - \alpha^3 - \alpha^2(\sqrt{\alpha^2 - 4\alpha h}) & -\frac{a}{c} - \frac{a}{c(\alpha c(\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-4\alpha h}}{2}(m-1)))} \\ 0 & \frac{\alpha b[\frac{\alpha}{2} - \frac{(\sqrt{\alpha^2-4\alpha h})}{2}](m-1)}{\alpha c[\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-4\alpha h}}{2}](m-1)-1} - d \end{bmatrix}$$

$$\alpha \geq 4h$$

from the equilibrium points that we found.

Boundary: P_1

Because the matrices for P_0 and P_1 are upper triangular, using the eigenvalues to determine behavior can be done easily.

Given P_1 is

$$\begin{bmatrix} (\alpha - \alpha^3 - \alpha^2(\sqrt{\alpha^2 - 4\alpha h}) & -\frac{a}{c} - \frac{a}{c(\alpha c(\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-4\alpha h}}{2}(m-1)))} \\ 0 & \frac{\alpha b[\frac{\alpha}{2} - \frac{(\sqrt{\alpha^2-4\alpha h})}{2}](m-1)}{\alpha c[\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-4\alpha h}}{2}](m-1)-1} - d \end{bmatrix}$$

The two eigenvalues are

$$\lambda_1 = \alpha - \alpha^3 - \alpha^2 \triangle$$

$$\lambda_2 = \frac{\alpha b[\frac{\alpha}{2} - \frac{(\sqrt{\alpha^2-4\alpha h})}{2}](m-1)}{\alpha c[\frac{\alpha}{2} - \frac{\sqrt{\alpha^2-4\alpha h}}{2}](m-1) - 1} - d$$

Boundary P_1 cont

$\triangle$ can range from 0 *to* α, dependent on h. Thus, λ_1 is necessarily positive if $\alpha < 1/4$, but may be positive even if $\alpha = 1$
λ_2 is complicated by the term $b - cd$ if $b < cd$, $\lambda_2 > 0$
but if $b > cd$, then $\lambda_2 > 0$ if $\lambda < \frac{-d}{(b-cd)\Theta n}$
where $\Theta = \alpha/2 - \triangle/2$ and $\triangle = \sqrt{\alpha^2 - 4\alpha h}$

Coexistence Point

At P_2, the determinant (D) and trace (T) were evaluated to be:

$$D = \alpha d(2x-1) - \frac{adyn}{z^2} - \frac{\alpha bx(2x-1)n}{z} \tag{4}$$

$$T = -\alpha(2x-1) - d + \frac{bxn}{z} - \frac{ayn}{z} + \frac{acxyn^2}{z^2} \tag{5}$$

where $z = cx(m-1) - 1$ and $n = m - 1$
Note that $z < n < 0$.
Complex series of conditions to determine the behavior of this point(explained in previous presentation).

Stable Focus

Since $T < 0$, $D < 0$ and $T^2 - 4D < 0$, P_2 is a stable focus.

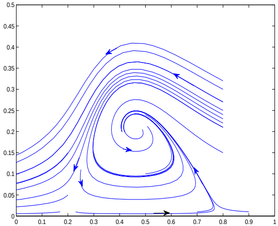

Figure: Stable Focus Phase Portrait

$$\alpha = .6 \ a = .6 \ b = .5 \ c = .1 \ d = .2 \ m = .1 \ h = .1$$
$$T = -.0575 \ D = .0204 \ T^2 - 4D = -.0783$$

$$(0.4630, 0.2049)$$

Stable Node

Since $T < 0$, $D > 0$ and $T^2 - 4D > 0$, P_2 is a stable node.

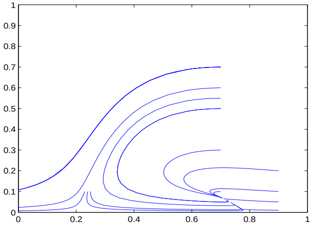

Figure: Stable Node Phase Portrait

$$\alpha = .6 \; a = .6 \; b = .5 \; c = .1 \; d = .3 \; m = .1 \; h = .1$$
$$T = -.2825 \; D = .0094 \; T^2 - 4D = .0421$$

$$(0.7092, 0.0659)$$

Bifurcations

Drastic change in qualitative behavior of solutions for a small change in one or more parameters

Can be (usually) detected using XPPAUT

Proven using Sotomayor's Theorem

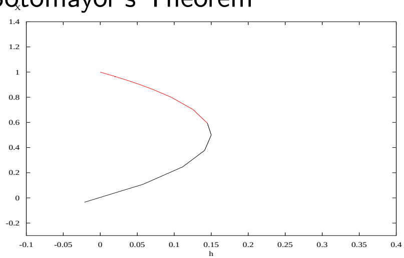

Figure: Saddle-Node Bifurcation Diagram

Sotomayor's Theorem

Theorem

Suppose that $f(x_0, \mu_0) = 0$ *and that the* $n \times n$ *matrix* $A \equiv Df(x_0, \mu_0)$ *has a simple eigenvalue* $\lambda = 0$ *with eigenvector* $\mathbf{v}$ *and that* A^T *has an eigenvector* $\mathbf{w}$ *corresponding to the eigenvalue* $\lambda = 0$*. Furthermore, suppose that* A *has* k *eigenvalues with negative real part and* $(n - k - 1)$ *eigenvalue with positive real part and that the following conditions are satisfied:*

$$\mathbf{w}^T f_\mu(x_0, \mu_0) \neq 0, \quad \mathbf{w}^T D^2 f(x_0, \mu_0)(\mathbf{v}, \mathbf{v}) \neq 0. \tag{6}$$

Then the system experiences a saddle-node bifurcation at the equilibrium point x_0 *as the parameter* μ *passes through the bifurcation value* $\mu = \mu_0$*.*

Application of Sotomayor's Theorem

Theorem

If $x = \frac{1}{2}$, $b \neq \frac{dz}{x(m-1)}$ and $\alpha \geq 4h$, then systems Model 1 and Model 2 undergoes a saddle-node bifurcation at $(\frac{1}{2}, 0)$.

Proof.

$$w = \begin{bmatrix} 1 \\ \frac{axn}{bxn-dz} \end{bmatrix} f_\mu(x_0, \mu_0) = \begin{bmatrix} -1 & 0 \end{bmatrix} \tag{7}$$

Thus

$$\mathbf{w}^T f_\mu(x_0, \mu_0) \neq 0 \tag{8}$$

$D^2 f(x_0)(v, v)$

$$D^2 f(x_0)(v,v) = \begin{bmatrix} \frac{\partial^2 f_1(x_0)}{\partial x^2} v_1 v_1 + \frac{\partial f_1^2(x_0)}{\partial x \partial y} v_1 v_2 + \frac{\partial f_1^2(x_0)}{\partial x \partial y} v_2 v_1 + \frac{\partial^2 f_1(x_0)}{\partial y^2} v_2 v_2 \\ \frac{\partial^2 f_2(x_0)}{\partial x^2} v_1 v_1 + \frac{\partial f_2^2(x_0)}{\partial x \partial y} v_1 v_2 + \frac{\partial f_2^2(x_0)}{\partial x \partial y} v_2 v_1 + \frac{\partial^2 f_2(x_0)}{\partial y^2} v_2 v_2 \end{bmatrix}$$

$$\mathbf{w}^T D^2 f(P_2)(v,v) = \mathbf{w}^T \begin{bmatrix} -2\alpha - \frac{an}{z^2}\left(\frac{axn}{bx(m-1)-dz}\right) \\ \\ \frac{b(1-m)}{z^2}\left(\frac{axn}{bxn-dz}\right) \end{bmatrix}$$

$$= -2\alpha + \frac{-an}{z^2}\left(\frac{axn}{bxn-dz}\right) - \frac{bn}{z^2}\left(\frac{axn}{bxn-dz}\right)^2 \neq 0$$

Other Bifurcations

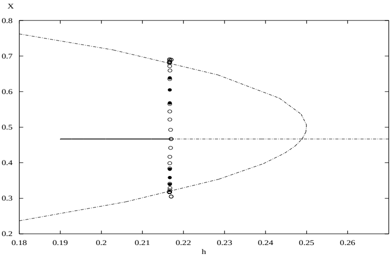

Figure: Saddle-Node, Transcritical and Hopf Bifurcations

Hopf Bifurcation

Theorem

Under the conditions for the coexistence equilibrium P_2 to be of center-type, there exists a Hopf bifurcation for the system.

Proof. For a system of the form:

$$\dot{x} = ax + by + p(x,y) \quad \text{and} \quad \dot{y} = cx + dy + q(x,y)$$

where
$p(x,y) = \sum a_{ij}x^i y^j =$
$(a_{20}x^2 + a_{11}xy + a_{02}y^2) + (a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3)$ and
$q(x,y) = \sum b_{ij}x^i y^j =$
$(b_{20}x^2 + b_{11}xy + b_{02}y^2) + (b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3)$
with $ad - bc > 0$ and $a + d = 0$.

First, we shift our equilibrium point of P_2 to the origin via the change in coordinates $\bar{x} = x - x^*$ and $\bar{y} = y - y^*$ and then we expand our expressions for $\bar{x}$ and $\bar{y}$ in a power series to get

$$\begin{aligned} \dot{x} &= \alpha(\bar{x} + x^*)(1 - (\bar{x} + x^*)) - \frac{a(1-m)(\bar{x}+x^*)(\bar{y}+y^*)}{1+c(1-m)(\bar{x}+x^*)} - h \\ \dot{y} &= -d(\bar{y} + y^*) + \frac{b(1-m)(\bar{x}+x^*)(\bar{y}+y^*)}{1+c(1-m)(\bar{x}+x^*)} \end{aligned} \quad (9)$$

Hopf Bifurcation at P_2

$a_{10} = \alpha(1-2x^*) - \frac{a(1-m)y^*}{w^3}$, $a_{01} = -\frac{a(1-m)x^*}{w}$, $a_{20} = -\alpha + \frac{ac(1-m)^2y^*}{w^3}$, $a_{11} = -\frac{a(1-m)}{w^2}$, $a_{02} = 0$, $a_{21} = \frac{ac(1-m)^2}{w^3}$, $a_{30} = -\frac{ac^2(1-m)^3y^*}{w^4}$, $b_{10} = \frac{b(1-m)y^*}{w^2}$,

$b_{01} = -d + \frac{b(1-m)x^*}{w}$, $b_{20} = -\frac{bc(1-m)^2y^*}{w^3}$, $b_{11} = \frac{b(1-m)}{w^2}$, $b_{02} = 0$, $b_{03} = 0$, $b_{12} = 0$, $b_{21} = -\frac{bc(1-m)^2}{w^3}$, $b_{30} = \frac{bc^2(1-m)^3y^*}{w^4}$
where $w = 1 + c(1-m)x^*$.

$$\sigma = 244.213 \neq 0$$

$$D = a_{10}b_{01} - a_{01}b_{10} = 0.00143 > 0$$

$$T = a_{10} + b_{01} = 3.61973 * 10^-10 \approx 0$$

P_2 is also of center-type.

Influence of Refuge on Coexistence

$$\frac{dx_3}{dm} = \frac{d(b+cd)}{[(b-cd)(1-m)]^2} > 0$$

&

$$\frac{dy_3}{dm} = \frac{bh}{ad(1-m)^3(b-cd)^2}[2\alpha d - (1-m)(b-\alpha cd] > 0$$

Conditions:

If $b > \alpha cd$, then $\frac{dy_3}{dm} > 0$ if $0 < m < m^*$ where $m^* = \frac{b-(c+2)\alpha d}{b-\alpha cd}$ or $\frac{dy_3}{dm} < 0$ if $m^* < m < 1$.

Else, if $b < \alpha cd$, then $m^* < m < 1$ and $\frac{dy_3}{dm} < 0$ and if $0 < m < m^*$ then $\frac{dy_3}{dm} > 0$.

Model 2: Threshold Harvesting

$$H_1(x) = \begin{cases} 0 & x < T_1 \\ \frac{h(x-T_1)}{T_2-T_1} & T_1 \leq x \leq T_2 \\ h & x > T_2 \end{cases} \tag{10}$$

Case 1: $H(x) = 0, x < T_1$

Equilibrium Points

$$Q_1 = (0,0), \quad Q_2 = (1,0)$$

$$Q_3 = \left(\frac{d}{(b-cd)(1-m)}, \frac{b}{a} \left(\frac{(b-cd)(1-m)-d}{(b-cd)^2(1-m)^2} \right) \right)$$

General Jacobian

$$\begin{bmatrix} 1-2x-\frac{ay(m-1)^2}{z^2}-\alpha(x-1)-\frac{ay(m-1)}{z}-\alpha x & -\frac{ax(m-1)}{z} \\ \frac{by(m-1)}{z^2} & \frac{bx(m-1)}{z}-d \end{bmatrix}$$

The Jacobian of Q_1 is $J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -d \end{bmatrix}$ so Q_1 is a saddle.

The Jacobian of Q_2 is

$$J(1,0) = \begin{bmatrix} -1 & -\frac{a(1-m)}{1+c(1-m)} \\ 0 & -d + \frac{b(1-m)}{1+c(1-m)} \end{bmatrix}$$

Conditions:
(a) Q_2 is a saddle if $(1-m)b > [1+(1-m)c]d$.
(b) Q_2 is a stable node if $(1-m)b < [1+(1-m)c]d$.
(c) Q_2 is never a focus or center type.

The Jacobian at Q_3 is

$$J(x_3, y_3) = \begin{bmatrix} \frac{d[b-cd)(c(1-m)-1)-2cd]}{b(b-cd)(1-m)} & -\frac{ad}{b} \\ \frac{(b-cd)(1-m)-d}{a(1-m)} & 0 \end{bmatrix}$$

Using Trace Determinant Analysis, $D = \frac{d[(b-cd)(1-m)-d]}{b(1-m)}$,

$T^2 - 4D = \frac{d}{b^2(1-m)^2}\left[\frac{d[-(b-cd)-c(b-cd)(1-m)^2]}{(b-cd)^2} - 4b(1-m)[(b-cd)(1-m)-d]\right]$

$$D > 0$$

given the conditions that $b - cd \quad \text{and} \quad d < (b-cd)(1-m)$

$$T = \frac{d[b - cd)(c(1 - m) - 1) - 2cd]}{b(b - cd)(1 - m)}$$

- Q_3 can never be a saddle.
- Q_3 is a node if $d[-(b - cd) - c(b - cd)(1 - m)]^2 \geq 4b(b - cd)^2(1 - m)[(b - cd)(1 - m) - d]$.
  -If $(b - cd)[c(1 - m) - 1] < 2cd$, then the node is stable, and unstable if the inequality is reverse.
- Q_3 is a focus if $d[-(b - cd) - c(b - cd)(1 - m)]^2 < 4b(b - cd)^2(1 - m)[(b - cd)(1 - m) - d]$.
  - If $(b - cd)[c(1 - m) - 1] < 2cd$, then the focus is stable, and unstable if the inequality is reverse.
- Q_3 is of center-type if $(b - cd)[c(1 - m) - 1] = 2cd$.

Case 2: $H(x) = \frac{h(x-T_1)}{T_2-T_1}$, $T_1 \le x \le T_2$

Three Equilibrium Points

$$R_1 = \left(\frac{h-\alpha(T_1-T_2)-\sigma}{2\alpha(T_2-T_1)}, 0\right), \quad R_2 = \left(\frac{-h+\alpha(T_1-T_2)-\sigma}{2\alpha(T_2-T_1)}, 0\right)$$

where $\sigma = \sqrt{4\alpha h T_1(T_1-T_2)+(h+\alpha(-T_1+T_2))^2}$ and $h > \alpha(T_1-T_2)+\sigma$.

$$R_3 = (x_3, y_3)$$

$$x_3 = \frac{d}{(b-cd)(1-m)}$$

$$y_3 = b\frac{-\alpha[b(m-1)+d(1+c(1-m))][-d(b-cd)(T_1-T_2)]+h(m-1)(-d-(b-cd)(m-1)T_1)(b-cd)^2}{[-d(b-cd)^3(T_1-T_2)]a(m-1)^2}$$

The general Jacobian is

$$\begin{bmatrix} \frac{h}{T_2 - T_1} + \alpha - 2\alpha x - \frac{ay(1-m)}{z^2} & -\frac{ax(1-m)}{z} \\ \frac{by(1-m)}{z^2} & -d - \frac{bx(1-m)}{z} \end{bmatrix}$$

The Jacobian at R_1 is

$$\begin{bmatrix} \frac{\sigma}{T_2 - T_1} & \frac{2\alpha h T_1(m-1)}{h+2chT_1(1-m)-\alpha(T_1-T_2)+\sigma} \\ 0 & -d - \frac{2bhT_1(m-1)}{h+2chT_1(1-m)-\alpha(T_1-T_2)+\sigma} \end{bmatrix}$$

and at R_2

$$\begin{bmatrix} \frac{-2h-2\alpha(T_1-T_2)+\sigma}{T_1-T_2} & \frac{2ahT_1(m-1)}{-h+2chT_1(1-m)+\alpha(T_1-T_2)+\sigma} \\ 0 & -d + \frac{2bhT_1(m-1)}{h+2chT_1(m-1)-\Phi+\alpha(T_2-T_1)} \end{bmatrix}$$

$$R_1 : \lambda_1 = \frac{\sigma}{T_2 - T_1}, \quad \lambda_2 = -d - \frac{2bhT_1(m-1)}{h+2chT_1(1-m)-\alpha(T_1-T_2)+\sigma}$$

$$R_2 : \lambda_1 = \frac{-2h+2\alpha(T_2-T_1)+\sigma}{T_1-T_2}, \quad \lambda_2 = -d + \frac{2bhT_1(m-1)}{h+2chT_1(m-1)-\Phi+\alpha(T_2-T_1)}$$

where $\Phi = \sqrt{h^2 + 2\alpha h(2T_1 - 1)(T_1 - T_2) + \alpha^2(T_1 - T_2)^2}$

R_1: never be of center-type or focus.
R_2: if $T_1 < \frac{1}{2}$ then R_2 will also be real (saddle or node).
There was a saddle-node bifurcation at R_1, the proof is similar to the previous one with Sotomayor's Theorem.

The Jacobian at R_3 is

$$\begin{bmatrix} \alpha - \frac{2\alpha d}{(b-cd)(m-1)} + b\left(\frac{(-\Omega+(b-cd)h(d+(b-cd)(m-1)^2T_1))}{(m-1)[d(T_1-T_2)](b-2cd)^2}\right) & \frac{ad}{b-2cd} \\ b^2\left(\frac{(\Omega+d(-b+cd)h((b-cd)^2)h(m-1)^2T_1)}{(m-1)[d(T_1-T_2)](b-2cd)^2}\right) & \frac{-2d(b-cd)}{b-2cd} \end{bmatrix}$$

where

$$\Omega = \alpha[b(m-1) + d(1 + c(1-m))[d(T_1 - T_2)]$$

Case 3. $H(x) = h, x > T_2$

Since rate of harvesting is constant, the behavior of this case will be similar to case 1.

No Periodic Solution for Linear Harvesting

Proof. First, we shift the equilibrium points to the origin by using a change in variables. $v = a(1-m)y$,
$dt = [1 + cx(1-m)]ds, \dot{v} = a(1-m)\dot{y}, \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds}$ and $\frac{dv}{ds} = \frac{dv}{dt}\frac{dt}{ds}$

$$\frac{dx}{ds} = \alpha x(1-x)[1+cx(1-m)] - xv - \frac{h(x-T_1)}{T_2-T_1}[1+cx(1-m)] = F_1$$

&

$$\frac{dv}{ds} = xv[(b-cd)(1-m)] - dv = F_2$$

Replacing v with y and $R = \frac{1}{xy}$, we have

$$RF_1 = \frac{1}{y}[\alpha(1+cx(1-m) - x - cx^2(1-m))] - 1 - \frac{h}{(T_2-T_1)y}\left[\frac{x-T_1}{x}\right]$$

&

$$RF_2 = (b-cd)(1-m) - \frac{d}{x}$$

Then,

$$\frac{\partial(RF_1)}{\partial x} = \frac{1}{y}[\alpha[c(1-m)-1]-2\alpha cx(1-m)] - \frac{hT_1}{(T_2-T_1)x^2y}$$

&

$$\frac{\partial(RF_2)}{\partial y} = 0$$

Hence if $c(1-m) < 1$, $\frac{\partial(RF_1)}{\partial x} + \frac{\partial(RF_2)}{\partial y} < 0 \ \forall x, y > 0$ which indicates that there are no periodic solutions by Dulac's Criterion.

3 Dimensional Model

$$\begin{aligned}\dot{x} &= \alpha x(1-x) - \frac{a(1-m)xy}{1+c(1-m)x} + D_1(v-x) \\ \dot{v} &= -d_v v + D_2(x-v) \\ \dot{y} &= -dy + \frac{b(1-m)x_1 y}{1+c(1-m)x_1 y}\end{aligned} \quad (11)$$

Dispersal equation - some of the prey (v) is inaccessible to predator

Equilibria

3 Equilibria:

$$M_0 = (0, 0, 0)$$

$$M_1 = \left(\frac{D_2 v + d_v v}{D_2}, \frac{D_2(-D_1 d_v + D_2\alpha + d_v\alpha)}{(D_2 + d_v)^2\alpha}, 0\right)$$

$$M_2 = \left(\frac{d}{(b-cd)(1-m)}, \frac{-dD_2}{(b-cd)(D_2+d_v)(m-1)}, \gamma\right)$$

where

$$\gamma = \frac{b(b-cd)D_1 d_v(m-1) - b(D_2+d_v)(b(m-1)+d(1+c-cm))\alpha}{a(b-cd)^2(D_2+d_v)(m-1)^2}$$

M_0

Conditions on M_0:
if $D_1 d_v < \alpha(D_1 + d_v)$ then M_0 is a saddle
if $\alpha > D_1 + d_v + D_2$ then the point is stable
Cannot be a focus or center, if not a saddle, it is a node:
Biologically reasonable

Boundary Equilibrium: M_1

It can be a saddle under some conditions, otherwise it is a node(stable or unstable).
Not a focus or center

Node

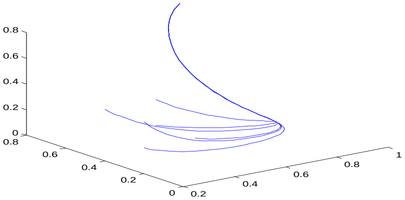

Figure: Boundary Node: $(.9159, .4585, 0.000)$

$$\alpha = .6 \ a = .6 \ b = .5 \ c = .1 \ d = .5 \ m = .1 \ D_1 = .1 \ D_2 = .1$$
$$d_v = .1$$

Orbits with $y = 0$

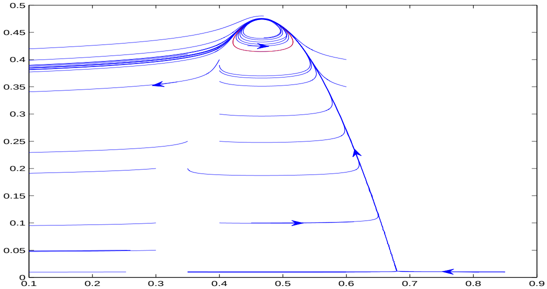

Figure: Orbit in Red

M_2: Coexistence

Incredibly complex system of conditions governing local behavior(Conditions on the sign of the real part of eigenvalues, or Trace Determinant Expressions)
Numerical simulations were obtained

M_2: Focus

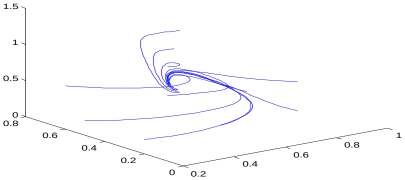

Figure: Coexistence Focus: (.2264, .1134, .7828)

$$\alpha = .6 \ a = .6 \ b = .5 \ c = .1 \ d = .1 \ m = .1 \ D_1 = .1 \ D_2 = .1 \ d_v = .1$$

Double Transcritical Bifurcation

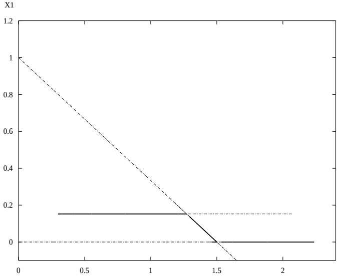

Figure: 2 Transcritical Bifurcations

$$a = .1, \alpha = 1, c = .2, d = .1, d_v = .4, D_2 = .2, m = .1$$

Sotomayor's Theorem was used to prove the result of XPPAUT, but the system was too complicated to effectively set conditions for $\lambda = 0$

3d System: Harvesting

A new 3-Dimensional system includes the harvesting function $H(x) = h$ in the prey equation. Thus:

$$\begin{aligned} \dot{x} &= \alpha x(1-x) - \frac{a(1-m)xy}{1+c(1-m)x} - D_1(v-x) - h \\ \dot{v} &= d_v v + D_2(x-v) \\ \dot{y} &= -dy + \frac{b(1-m)xy}{1+c(1-m)x} \end{aligned} \tag{12}$$

Orbit

With this modification, periodic orbits and Hopf bifurcation appear.

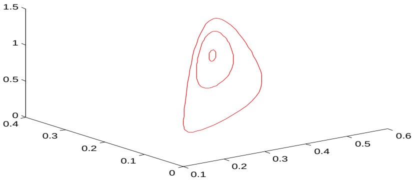

Figure: 3 Orbits

$$\alpha = 1 \; a = .2 \; b = 1 \; c = .2 \; h = .11 \; m = .3 \; D_1 = .4 \; D_2 = .3 \; d_v = .3$$

Hopf Bifurcation

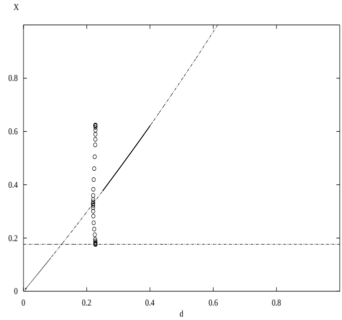

Figure: Hopf Bifurcation Diagram

$$\alpha = 1 \; a = .2 \; b = 1 \; c = .2 \; h = .11 \; m = .3 \; D_1 = .4 \; D_2 = .3 \; d_v = .3$$

Conclusions

- Studied three models, with different harvesting functions and refuge
- 3-D model included dispersal of prey, in two different habitats
- Local stability was analyzed
- Existence of saddle-node and Hopf bifurcations was proved
- Other bifurcations were numerically detected
- Unstable periodic solutions were computed
- Non-existence of periodic solutions under certain conditions was proved
- Influence of refuge on prey density was analyzed