Modeling and Analysis of Anaerobic Digestion in a Bioreactor

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Advantages of Wastewater Treatment

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- Produce biogas

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Degradation Process

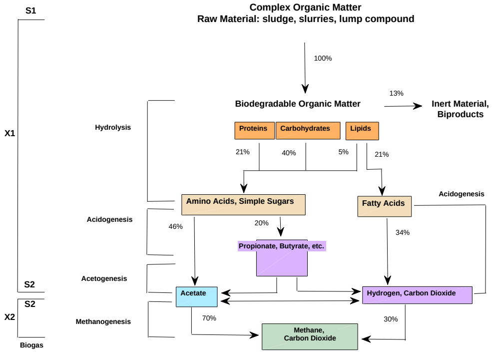

Figure : Detailed Flowchart of Model for Biogas Production

Condensed 4-Dimensional (two-step reaction process) System

Definition

4-Dimensional System:

$$\begin{aligned} S_1' &= -X_1 k_1(\mu_1(S_1)) + D(S_{1in} - S_1) \\ X_1' &= X_1(\mu_1(S_1) - D\alpha) \\ S_2' &= D(S_{2in} - S_2) + X_1 k_2(\mu_1(S_1)) - X_2 k_3(\mu_2(S_2)) \\ X_2' &= X_2(\mu_2(S_2) - D\alpha) \end{aligned} \quad (1)$$

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- D: Dilution factor for incoming and outgoing substrate and bacteria.
- $\mu_1(S_1)$ and $\mu_2(S_2)$ are functions used to demonstrate the growth of bacteria 1 and 2, respectively.

Simplified Two-Step Reaction Process

Definition

Two Steps:

$$\begin{aligned} \text{Acidogenesis}: \quad & k_1 S_1 \xrightarrow{\mu_1(S_1)X_1} X_1 + k_2 S_2 \\ \text{Methanogenesis}: \quad & k_3 S_2 \xrightarrow{\mu_2(S_2)X_2} X_2 + k_4 CH_4 \end{aligned} \tag{2}$$

Acidogenesis: Organic substrate (S_1) is broken down into volatile fatty acids (S_2) by acidogenic bacteria (X_1).

Methanogenesis: Volatile fatty acids (S_2) are degraded to produce CH_4 and CO_2 by methanogenic bacteria (X_2).

Concerns Regarding Approach to Attaining Steady-State

Two Main Concerns:

- Growth of Bacteria

To attain a steady-state: The substrate flow and gas production must remain constant and continuous. The growth requirements for bacteria must remain constant over time.

Definition

4-Dimensional System:

$$\begin{aligned} 0 &= -X_1 k_1(\mu_1(S_1)) + D(S_1 in - S_1) \\ 0 &= X_1(\mu_1(S_1) - D\alpha) \\ 0 &= D(S_2 in - S_2) + X_1 k_2(\mu_1(S_1)) - X_2 k_3(\mu_2(S_2)) \\ 0 &= X_2(\mu_2(S_2) - D\alpha) \end{aligned} \tag{3}$$

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Steady-State

Substrate Balance: $\frac{dS}{dt} = DS_0 - DS + \frac{dS}{dt}$

Bacteria Balance: $\frac{dX}{dt} = DX_0 - DX + \mu(S)X + kdX$

Equilibrium point: $\frac{dX}{dt} = 0 \quad \frac{dS}{dt} = 0$ as $t \longrightarrow \infty$

$\frac{dS}{dt}$ and $\frac{dX}{dt}$: Accumulation

DS_0 and DX_0: Diluted Input $\quad$ DS and DX: Diluted Output

Different Approaches

Each differential system under study was characterized by its unique combination of two of the numerous hypothesized bacterial growth functions; our study included application of the Monod and Haldane functions of bacteria growth [d],

Monod: $\mu_1(S_1) = \frac{m_1 S_1}{K_1 + S_1}$

Haldane: $\mu_2(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_I}}$

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- K_1 and K_2 : Substrate Concentrations at 50 percent of maximum specific growth rate(see graph).
- K_I : Substrate concentration where bacteria growth is reduced to 50 percent of it's maximum growth rate due to substrate inhibition (see graph).

Bacteria Growth Kinetics

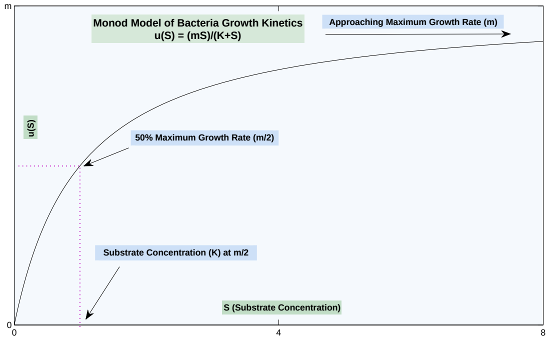

Figure : Monod Model for Bacteria Growth Kinetics

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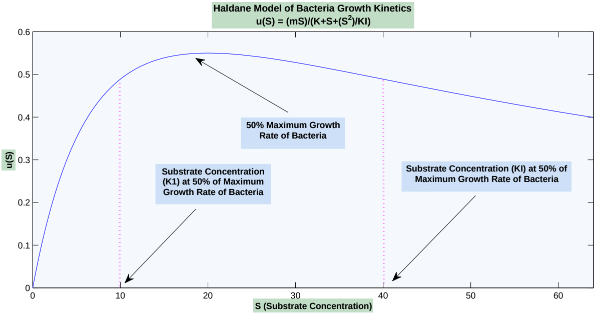

Figure : Haldane Model for Bacteria Growth Kinetics

Theory Behind Four-Dimensional Systems A and B

Two Hypotheses:

- 4-Dimensional System A: The growth rates of X_1 and X_2 are both increasing functions of added substrate (S_1 and S_2).

$$\mu_1(S_1) = \frac{m_1 S_1}{K_1 + S_1} \qquad \mu_2(S_2) = \frac{m_2 S_2}{K_2 + S_2}$$

- 4-Dimensional System B: The growth rate of X_1 is an increasing function of substrate (S_1) and the growth rate of X_2 approaches a maximum at a medium substrate concentration (K_I = medium S_2 concentration).

$$\mu_1(S_1) = \frac{m_1 S_1}{K_1 + S_1} \qquad \mu_2(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_I}}$$

Introduction of Foreign Toxin

$$
\begin{aligned}
S_1' &= -X_1 k_1 e^{-y\mu}(\mu_1(S_1)) + D(S_{1in} - S_1) \\
X_1' &= X_1(e^{-y\mu}\mu_1(S_1) - D\alpha) \\
S_2' &= D(S_{2in} - S_2) + X_1 k_2 e^{-y\mu}(\mu_1(S_1)) - X_2 k_3(\mu_2(S_2)) \\
X_2' &= X_2(\mu_2(S_2) - D\alpha) \\
y' &= D(y_{in} - y) - X_2 k_4 \mu_3(y)
\end{aligned} \tag{4}
$$

where y represents the toxin.

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Types of Inhibition

- Competitive Inhibition: A foreign species similar in structure to the substrate binds to the enzymes, inhibiting reaction spots.

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- Uncompetitive Inhibition: A foreign species not necessarily similar in structure to the substrate binds to the enzyme-substrate complexes, preventing completion of the reaction.

Effect of a Toxin

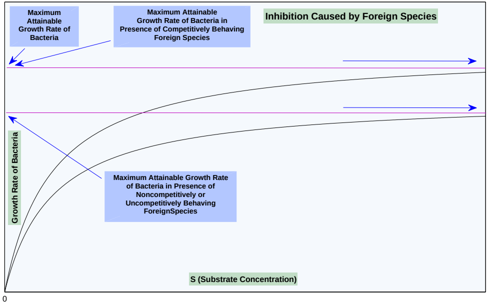

Figure : Illustrations of Competition by Non-Substrate Species

Five-Dimensional Systems

The four systems differed amongst each other in terms of the models used to represent X_2 growth and toxin consumption.

Four Hypotheses:

- 5-Dimensional System A: The Monod model is used to represent the growth rate of X_1, the Haldane model is used to represent the growth rate of X_2, and the Monod model is used to represent the consumption rate of the toxin.

$$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1} \quad \mu(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_{I1}}} \quad \mu(y) = \frac{m_4 y}{K_4 + y}$$

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- 5-Dimensional System B: The Monod model is used to represent the growth rate of X_1, the Haldane model is used to represent both the growth rate of X_2 and the consumption rate of the toxin.

$$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1} \quad \mu(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_{I1}}} \quad \mu(y) = \frac{m_4 y}{K_4 + y + \frac{y^2}{K_{I2}}}$$

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Four Hypotheses:

- 5-Dimensional System C: The Monod model is used to represent the growth rate of X_1, the growth rate of X_2, and the consumption rate of the toxin.

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- 5-Dimensional System D: The Monod model is used to represent the growth rate of Bacteria 1 and 2, and the Haldane model is used to represent the consumption rate of the toxin.

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Six-Dimensional System

$$
\begin{array}{rcl}
S_1' & = & -X_1 k_1(\mu_1(S_1)) + D(S_1(1 + (rv)) - S_1) \\
X_1' & = & X_1(P(\mu_1(S_1) - D\alpha)) \\
S_2' & = & D(0 - S_2) + X_1 k_2(\mu_1(S_1)) - X_2 k_3(\mu_2(S_2)) \\
X_2' & = & X_2(\mu_2(S_2) - D\alpha) \\
u' & = & u(1 - u^2 - v^2) - 2\pi v \\
v' & = & v(1 - u^{)} - v^2) + 2\pi u
\end{array}
\qquad (5)
$$

where $S_1(1 + (rv)) = S_1(t) = S_1(1 + (r\sin(2\pi t)))$ and $0 \leq r \leq 1$.

Solving the 4-Dimensional Systems

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$$x_0 = (S_1, X_1, S_2, X_2)$$

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- Verify Algebraically Determined Discoveries with Illustrations of Behavior

Possible Results

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  All eigenvalues are nonzero values, nor are any of them purely imaginary.

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Types of Bifurcations

Bifurcation Value = Parameter Value at which Bifurcation Occurs
(Dilution Factor, D, acted as the variable parameter value)

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  Change in number of equilibrium points from one to three.

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- Hopf Bifurcation

  Periodic orbits arise from an equilibrium point as it changes stability at bifurcation value.

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- Precise solutions for equilibria were not found algebraically.
- Investigation carried out by analysis of the systems using XPPAUT.
- Bifurcations identified and verified algebraically using Sotomayor's Theorem, then further analyzed using MATLAB R2012b when necessary.
- Wolfram Mathematica 9.0 used to find any equilibria in the models, behavior determination of equilibria using MATLAB R2012b.

Solving the 5 and 6 Dimensional Systems

- When linearizing a six-dimensional system around a periodic orbit of period τ, a total of six Floquet multipliers are solved for from a 6 x 6 Monodromy matrix:

$$\lambda_i, 1 < i < 6$$

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- Using XPPAUT, find the number of stable Floquet multipliers corresponding to each periodic orbit of interest.
- $x' = f(x, \lambda) \quad \rightarrow \quad x' = A(t)x \qquad M = A(\tau)$

$$\textit{Solution: } x(t) = x(t+\tau), \textit{ for all } t \in \mathbb{R}.$$

Solving the 5 and 6 Dimensional Systems

- Hyperbolic periodic orbit:
  Exactly one Floquet multiplier must be equal to one.

  Stable Hyperbolic: Remaining Floquet multipliers are less than one.
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- Nonhyperbolic periodic orbit:
  More than one Floquet multiplier is located on the unit circle.

Results from 4-Dimensional System A

Four equilibria were found:

Equilibrium 1: $(S_{1in}, 0, S_{2in}, 0)$

Always Exists

Results from 4-Dimensional System A

Four equilibria were found:

Equilibrium 2: $(S_{1in}, 0, S_2^*(D), X_2^*(D))$

$$S_2^*(D) = \frac{DK_2\alpha}{m_2 - D\alpha} \qquad X_2^*(D) = \frac{1}{k_3\alpha}(S_{2in} - \frac{DK_2\alpha}{m_2 - D\alpha})$$

Conditions that must hold for point to exist:

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- $S_{2in} \geq (\frac{DK_2\alpha}{m_2 - D\alpha})$

Results from 4-Dimensional System A

Four equilibria were found:

Equilibrium 3: $(S_1^*(D), X_1^*(D), S_{2in}^*(D), 0)$

$S_1^*(D) = \frac{DK_1\alpha}{m_1 - D\alpha}$ $X_1^*(D) = \frac{1}{k_1\alpha}(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})$

$S_{2in}^*(D) = S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})$

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Results from 4-Dimensional System A

Four equilibria were found:

Equilibrium 4: $(S_1^*(D),\ X_1^*(D),\ S_2^*(D), X_{2in}^*(D))$

$$S_1^*(D) = \frac{DK_1\alpha}{m_1 - D\alpha} \quad X_1^*(D) = \frac{1}{k_1\alpha}(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha}) \quad S_2^*(D) = \frac{DK_2\alpha}{m_2 - D\alpha}$$

$$X_{2in}^*(D) = \frac{1}{k_3\alpha}[S_{2in} - \frac{DK_2\alpha}{m_2 - D\alpha} + (\frac{k_2}{k_1})(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})]$$

Results from 4-Dimensional System A

Conditions that must hold for point to exist (Equilibrium Point 4):

$m_1 > D\alpha \qquad m_2 > D\alpha \qquad S_{1in} \geq (\frac{DK_1\alpha}{m_1 - D\alpha})$

$S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha}) \geq (\frac{DK_2\alpha}{m_2 - D\alpha})$

Results from 4-Dimensional System A

Equilibrium Point 1

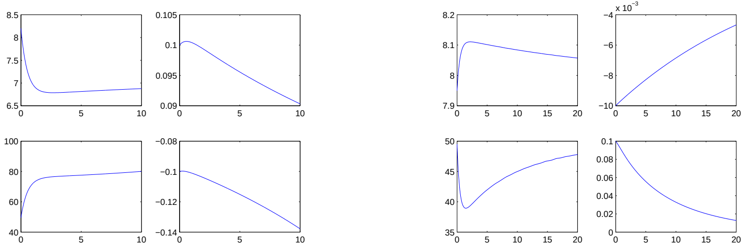


Figure : Case 8: Saddle (left) and Case 9: Stable Node (right). In Case 8, solutions are moving away from (8, 0, 50, 0) as $t \longrightarrow \infty$. In Case 9, solutions are moving towards (8, 0, 50, 0) as $t \longrightarrow \infty$.

Results from 4-Dimensional System A

Equilibrium Point 2

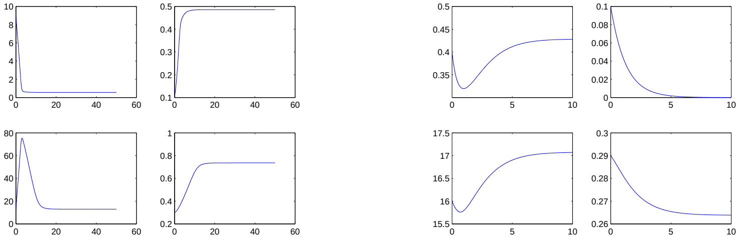


Figure : Case 7: Saddle (left) and Case 11: Stable Node (right). In Case 7, solutions are moving away from (9.0811, 0, 12.9454, 0.28312) as $t \longrightarrow \infty$. In Case 11, solutions are moving towards (0.4285, 0, 17.0765, 0.2638) as $t \longrightarrow \infty$.

Results from 4-Dimensional System A

Equilibrium Point 3

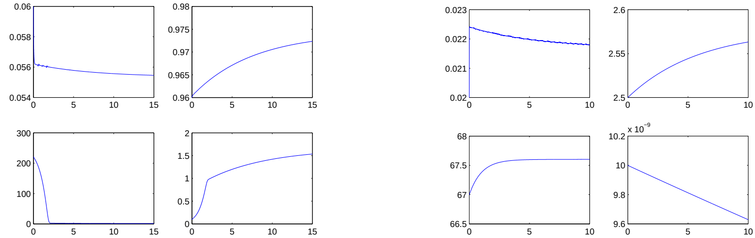


Figure : Case 7: Saddle (left) and Case 11: Stable Node (right). In Case 7, solutions are moving away from (0.0554, 0.9742, 221.273, 0) as $t \longrightarrow \infty$. In Case 11, solutions are moving towards 0.02167, 2.577, 67.603, 0) as $t \longrightarrow \infty$.

Results from 4-Dimensional System A

Equilibrium Point 4

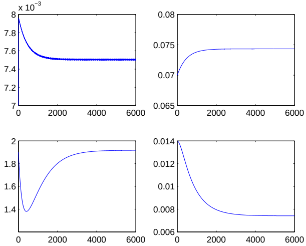


Figure : Case 4: Stable Node (left) and Case 5: Stable Node (right). In Case 4, solutions are moving towards (0.0076, 0.0743, 1.9314, 0.00735) as $t \longrightarrow \infty$. In Case 5, solutions are moving towards (0.4623, 0.5108, 37.5477, 0.00667) as $t \longrightarrow \infty$.

Set of Parameter Values used to Solve 4-Dimensional Systems

m_1	m_2	k_1	k_2	k_3	α
1.2	1.1	25	250	268	0.5
K_1	K_2	K_I	S_{1in}	S_{2in}	D
2	10	40	8	50	*variable*

Table : Parameter values used to solve 4-Dimensional Systems

Bifurcations found for 4-Dimensional System A

Equilibrium Point 1 $(S_{1in}, 0, S_{2in}, 0)$:

- Proved Transcritical bifurcation exists when

$D = D_1^* = (\frac{1}{\alpha})(\frac{m_2 S_{2in}}{K_2 + S_{2in}})$ or when

$D = D_2^* = (\frac{1}{\alpha})(\frac{m_1 S_{1in}}{K_1 + S_{1in}})$

When given parameter values are substituted into system, $D_1^* = \frac{11}{6}$ and $D_2^* = 1.92$.

Bifurcations found for 4-Dimensional System A

Equilibrium Point 2: $(S_{1in}, 0, S_2^*(D), X_2^*(D))$:

- Proved Transcritical bifurcation exists when

$$D = D^* = (\frac{1}{\alpha})(\frac{m_1 S_{1in}}{K_1 + S_{1in}})$$

When the following parameter values are substituted into system, $D^* = 0.1173$.

Randomly generated parameter values using MATLAB:

$m_1 = 1.3038,\ k_2 = 168.9636,\ K_1 = 2.3323,\ m_2 = 0.0253,$
$\alpha = 0.1739,\ K_2 = 19.892,\ S_{1in} = 9.7551,\ S_{2in} = 83.0481,$
$k_1 = 33.5826$

Bifurcations found for 4-Dimensional System A

Equilibrium Point 3 $(S_1^*(D), X_1^*(D), S_{2in}^*(D), 0)$:

- Proved Transcritical bifurcation exists when

$$D = D^* = (\frac{1}{\alpha})(\frac{m_2 q_1}{K_2 + q_1}) \qquad \text{such that}$$

$$q_1 = S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})$$

When given parameter values are substituted into system, $D^* = 1.877$.

Bifurcations found for 4-Dimensional System A

Equilibrium Point 4 $(S_1^*(D), X_1^*(D), S_2^*(D), X_{2in}^*(D))$:

- Proved Transcritical bifurcation exists when

$$D = D^* = (\frac{1}{K_2\alpha})(S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{K_1 D\alpha}{m_1 - D\alpha}))(m_2 - D\alpha)$$

such that

$$S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{K_1 D\alpha}{m_1 - D\alpha}) = \frac{K_2 D\alpha}{m_2 - D\alpha}$$

Bifurcations found for 4-Dimensional System A

Equilibrium Point 4 $(S_1^*(D), X_1^*(D), S_2^*(D), X_{2in}^*(D))$:

- Proved Transcritical bifurcation exists when

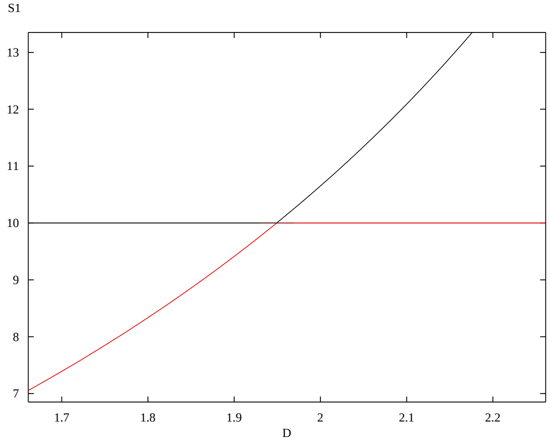
  $$D = D^* = (\frac{1}{K_2\alpha})(S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{K_1 D\alpha}{m_1 - D\alpha}))(m_2 - D\alpha)$$

  such that

  $$S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{K_1 D\alpha}{m_1 - D\alpha}) = \frac{K_2 D\alpha}{m_2 - D\alpha}$$

- A Hopf Bifurcation test was performed algebraically and using MATLAB, but no results were generated.

Bifurcations found for 4-Dimensional System A



Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 1: $(S_{1in}, 0, S_{2in}, 0)$

Always Exists

Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 2: $(S_1^*(D), X_1^*(D), S_{2in}^*(D), 0)$

$$S_1^*(D) = \frac{DK_1\alpha}{m_1 - D\alpha} \qquad X_1^*(D) = \frac{1}{k_1\alpha}\left(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha}\right)$$

$$S_{2in}^*(D) = S_{2in} + X_1^*(D)k_2\alpha$$

Conditions that must hold for point to exist:

- $m_1 > D\alpha$

Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 2: $(S_1^*(D), X_1^*(D), S_{2in}^*(D), 0)$

$S_1^*(D) = \frac{DK_1\alpha}{m_1 - D\alpha}$ $\quad X_1^*(D) = \frac{1}{k_1\alpha}(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})$

$S_{2in}^*(D) = S_{2in} + X_1^*(D)k_2\alpha$

Conditions that must hold for point to exist:

- $m_1 > D\alpha$
- $S_{1in} \geq (\frac{DK_1\alpha}{m_1 - D\alpha})$

Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 2: $(S_1^*(D), X_1^*(D), S_{2in}^*(D), 0)$

$$S_1^*(D) = \frac{DK_1\alpha}{m_1 - D\alpha} \qquad X_1^*(D) = \frac{1}{k_1\alpha}(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})$$

$$S_{2in}^*(D) = S_{2in} + X_1^*(D)k_2\alpha$$

Conditions that must hold for point to exist:

- $m_1 > D\alpha$
- $S_{1in} \geq (\frac{DK_1\alpha}{m_1 - D\alpha})$
- $S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha}) \geq 0$

Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 3: $(S_{1in}, 0, S_2^{1*}(D), X_2^1(D))$

$$S_2^{1*}(D) = (\frac{K_I}{2y})[(1-y) + ((1-y)^2 - (\frac{4K_2}{K_I})(y^2))^{1/2}]$$

$$X_2^1(D) = \frac{1}{k_3\alpha}(S_{2in} - S_2^{1*}(D))$$

$$y = \frac{D\alpha}{m_2}$$

Conditions that must hold for point to exist:
Amputate my foot...

Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 4: $(S_{1in}, 0, S_2^{2*}(D), X_2^2(D))$

$$S_2^{2*}(D) = (\frac{K_I}{2y})[(1-y) - ((1-y)^2 - (\frac{4K_2}{K_I})(y^2))^{1/2}]$$

$$X_2^2(D) = \frac{1}{k_3\alpha}(S_{2in} - S_2^{2*}(D))$$

$$y = \frac{D\alpha}{m_2}$$

Conditions that must hold for point to exist:
End world hunger...

Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 5: $(S_1^*(D), X_1^*(D), S_2^{1*}(D), X_2^{1*}(D))$

$S_1^*(D) = \frac{DK_1\alpha}{m_1 - D\alpha}$ $X_1^*(D) = \frac{1}{k_1\alpha}(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})$

$S_2^{1*}(D) = (\frac{K_I}{2y})[(1-y) + ((1-y)^2 - (\frac{4K_2}{K_I})(y^2))^{1/2}]$

$X_2^{1*}(D) = (\frac{1}{k_3\alpha})(S_{2in}^*(D) - S_2^{1*}(D))$

$y = \frac{D\alpha}{m_2}$

Conditions that must hold for point to exist:
Give up first born son...

Results from 4-Dimensional System B

Up to six equilibria were found:

Equilibrium 6: $(S_1^*(D), X_1^*(D), S_2^{2*}(D), X_2^{2*}(D))$

$$S_1^*(D) = \frac{DK_1\alpha}{m_1 - D\alpha} \qquad X_1^*(D) = \frac{1}{k_1\alpha}(S_{1in} - \frac{DK_1\alpha}{m_1 - D\alpha})$$

$$S_2^{2*}(D) = (\frac{K_I}{2y})[(1-y) - ((1-y)^2 - (\frac{4K_2}{K_I})(y^2))^{1/2}]$$

$$X_2^{2*}(D) = (\frac{1}{k_3\alpha})(S_{2in}^*(D) - S_2^{2*}(D))$$

$$y = \frac{D\alpha}{m_2}$$

Conditions that must hold for point to exist:
Amputate my other foot...

Results from 4-Dimensional System B

Equilibrium Point 1

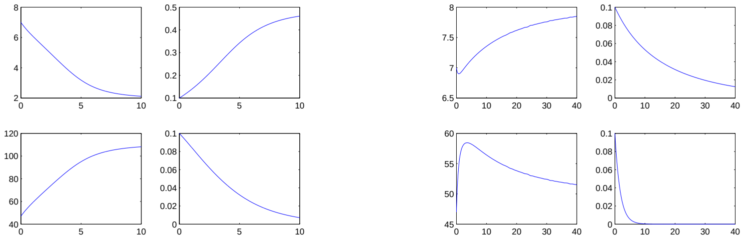


Figure : Case 5: Saddle (left) and Case 8: Stable Node (right). In Case 5, solutions are moving away from (8, 0, 50, 0) as $t \longrightarrow \infty$. In Case 8, solutions are moving towards (8, 0, 50, 0) as $t \longrightarrow \infty$.

Results from 4-Dimensional System B

Equilibrium Point 2

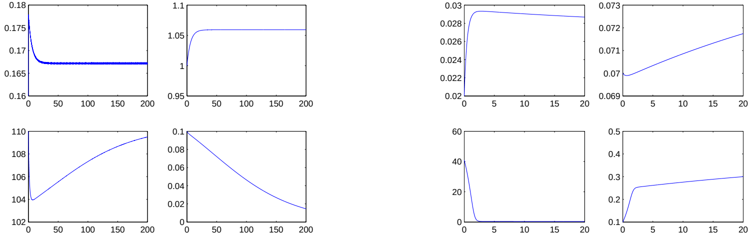


Figure : Case 2: Stable Node (left) and Case 8: Saddle (right). In Case 2, solutions are approaching (0.1672, 1.059, 110.474, 0) as $t \longrightarrow \infty$. In Case 8, solutions are moving away from (0.0272, 0.07557, 42.014, 0) as $t \longrightarrow \infty$.

Results from 4-Dimensional System B

Equilibrium Points 3 and 4

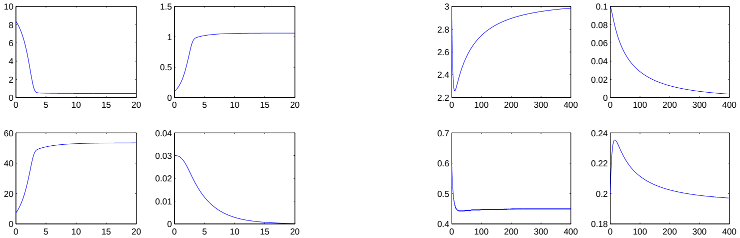


Figure : Equilibrium Point 3 Case A6: Saddle (left) and Equilibrium Point 4 Case C4: Stable Node (right). In EqPt 3 Case A6, solutions are moving away from (8.6266, 0, 7.08424, 0.03633) as $t \longrightarrow \infty$. In EqPt 4 Case C4, solutions are moving towards (3.0224, 0, 0.4495, 0.1948) as $t \longrightarrow \infty$.

Results from 4-Dimensional System B

Equilibrium Point 6

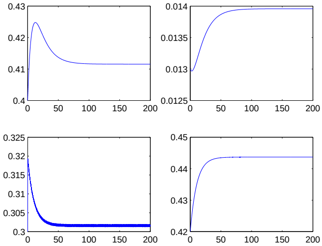


Figure : Equilibrium Point 6 Case B5: Stable Node. In EqPt 6 Case B5, solutions are moving towards (0.4116, 0.014, 0.3016, 0.4401) as $t \longrightarrow \infty$. No random parameters were generated that agreed with the predicted results from any cases of Equilibrium Point 5.

Bifurcations found from 4-Dimensional System B

Equilibrium Point 1 $(S_{1in}, 0, S_{2in}, 0)$:

- Proved Transcritical bifurcation exists when

  $D = D_1^* = (\frac{1}{\alpha})(\frac{m_1 S_{1in}}{K_1 + S_{1in}})$ or when

  $D = D_2^* = (\frac{1}{\alpha})(\frac{m_2 S_{2in}}{K_2 + S_{2in} + \frac{S_{2in}^2}{K_I}})$

  When given parameter values are substituted into system, $D_1^* = 1.92$ and $D_2^* = 0.898$.

Bifurcations found from 4-Dimensional System B

Equilibrium Point 1 $(S_{1in}, 0, S_{2in}, 0)$:

- Proved Transcritical bifurcation exists when

  $D = D_1^* = (\frac{1}{\alpha})(\frac{m_1 S_{1in}}{K_1 + S_{1in}})$ or when

  $D = D_2^* = (\frac{1}{\alpha})(\frac{m_2 S_{2in}}{K_2 + S_{2in} + \frac{S_{2in}^2}{K_I}})$

  When given parameter values are substituted into system, $D_1^* = 1.92$ and $D_2^* = 0.898$.
- A Pitchfork Bifurcation test was performed algebraically and using MATLAB in the case where $K_2 = \frac{S_{2in}^2}{K_I}$, but no results were generated.

Bifurcations found from 4-Dimensional System B

Equilibrium Point 2: $(S_1^*(D), X_1^*(D), S_{2in}^*(D), 0)$:

- Proved Transcritical bifurcation exists when

  $D = D^* = (\frac{1}{\alpha})(\frac{m_2 x}{K_2 + x + \frac{x^2}{K_I}})$ such that

  $x = S_{2in} + (\frac{k_2}{k_1})(S_{1in} - \frac{K_1 D\alpha}{m_1 - D\alpha})$

  When the following parameter values are substituted into system, $D^* = 0.4305$.

  Randomly generated parameter values using MATLAB:

  $$m_1 = 1.6362,\ k_2 = 250.9017,\ K_1 = 1.419,$$
  $$m_2 = 1.109,\ \alpha = 0.2349,\ K_2 = 3.7185,$$
  $$S_{1in} = 10.7751,\ S_{2in} = 66.2991,\ k_1 = 15.396,$$
  $$K_I = 24.1864,\ k_3 = 268.$$

A Pitchfork Bifurcation test was performed algebraically and using MATLAB in the case where $K_2 = \frac{x^2}{K_I}$, but no results were generated.

Bifurcations found from 4-Dimensional System B

Equilibrium Point 3: $(S_{1in}, 0, S_2^{1*}(D), X_2^1(D))$:

- Proved Transcritical bifurcation exists when

$D = D^* = (\frac{1}{\alpha})(\frac{m_1 S_{1in}}{K_1 + S_{1in}})$

When the following parameter values are substituted into system, $D^* = 0.4993$.

Randomly generated parameter values using MATLAB:

$m_1 = 0.6323$, $k_2 = 101.7029$, $K_1 = 1.6611$, $m_2 = 1.824$, $\alpha = 0.9722$, $K_2 = 8.0747$, $S_{1in} = 5.4899$, $S_{2in} = 77.3912$, $k_1 = 5.0144$, $k_3 = 268$, $K_I = 25.164$.

Bifurcations found from 4-Dimensional System B

Equilibrium Point 4: $(S_{1in}, 0, S_2^{2*}(D), X_2^2(D))$:

- Proved Transcritical bifurcation exists when

$D = D^* = (\frac{1}{\alpha})(\frac{m_1 S_{1in}}{K_1 + S_{1in}})$

When the following parameter values are substituted into system, $D^* = 0.2567$.

Randomly generated parameter values using MATLAB:

$m_1 = 0.2811$, $k_2 = 51.6183$, $K_1 = 1.0457$, $m_2 = 1.3676$, $\alpha = 0.9074$, $K_2 = 13.2164$, $S_{1in} = 5.0548$, $S_{2in} = 6.2729$, $k_1 = 8.4958$, $k_3 = 268$, $K_I = 49.88$.

Bifurcations found from 4-Dimensional System B

Equilibrium Point 5: $(S_1^*(D), X_1^*(D), S_2^{1*}(D), X_2^{1*}(D))$

Equilibrium Point 6: $(S_1^*(D), X_1^*(D), S_2^{2*}(D), X_2^{2*}(D))$

Hopf Bifurcation tests were performed for both equilibria, but no data was generated.

Results from 5-Dimensional System A

m_1	m_2	k_1	k_2	k_3	α	
1	3.2	30	215	100	0.3	
K_1	K_2	K_{I1}	K_{I2}	S_{1in}	S_{2in}	D
7	200	500	400	40	5	*variable*
m_4	k_4	K_4	y_{in}	μ		
1	5	1	5	0.06		

Table : Parameter values used for solving 5-Dimensional Systems

5-Dimensional System A

Combination of functions applied:

- Monod model for the growth of Bacteria 1

$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$

5-Dimensional System A

Combination of functions applied:

- Monod model for the growth of Bacteria 1

  $\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$

- Haldane model for the growth of Bacteria 2

  $\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_{I1}}}$

5-Dimensional System A

Combination of functions applied:

- Monod model for the growth of Bacteria 1

  $\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$
- Haldane model for the growth of Bacteria 2

  $\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_{I1}}}$
- Monod model for the consumptions of Toxin

  $\mu(y) = \frac{m_4 y}{K_4 + y}$

Bifurcations found when Solving 5-Dimensional System A

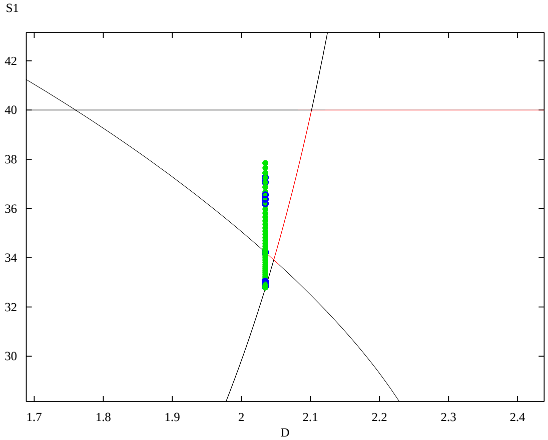


Figure : Illustration of Bifurcations found for 5 Dimensional System A

Stable Periodic Orbit of 5-Dimensional System A

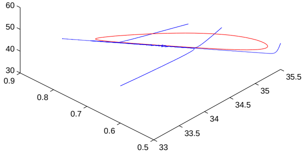


Figure : Presence of Stable Periodic Orbit in 5D System A. Though a Hopf bifurcation exists in this system from which stable periodic orbits are generated, the nonhyperbolic equilibrium point corresponding to the detected Hopf bifurcation displayed a negative solution for X_2. Unstable equilibria found: (40, 0, 5, 0, 5), (40, 0, 48.24, -1.44, -0.61). We graphed only 3-D projections of the solutions: S_1, X_1, and S_2.

Results from 5-Dimensional System B

Combination of functions applied:

- Monod model for the growth of Bacteria 1

$$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$$

Results from 5-Dimensional System B

Combination of functions applied:

- Monod model for the growth of Bacteria 1

$$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$$

- Haldane model for the growth of Bacteria 2

$$\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_{I1}}}$$

Results from 5-Dimensional System B

Combination of functions applied:

- Monod model for the growth of Bacteria 1

  $$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$$

- Haldane model for the growth of Bacteria 2

  $$\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_{I1}}}$$

- Haldane model for the consumptions of Toxin

  $$\mu(y) = \frac{m_4 y}{K_4 + y + \frac{y^2}{K_{I2}}}$$

Bifurcations found when Solving 5-Dimensional System B

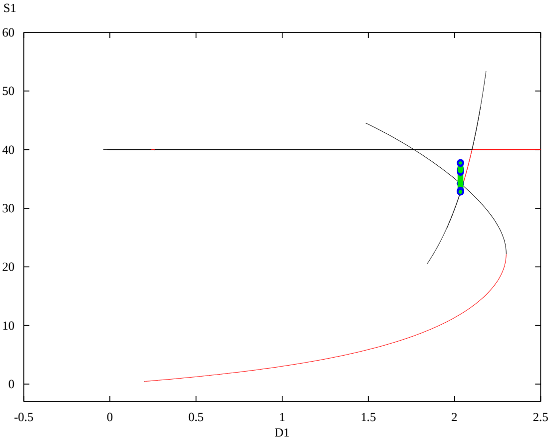
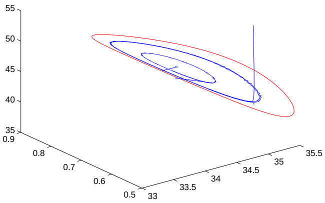


Figure : Illustration of Bifurcations found for 5 Dimensional System B

Stable Periodic Orbit of 5-Dimensional System B



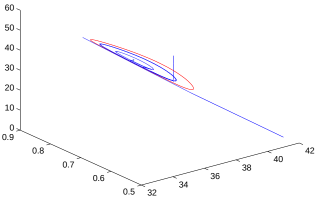


Figure : Proof of Stable Periodic Orbit in 5D System B, $D^* = 2.035$. Though a Hopf bifurcation exists in this system from which stable periodic orbits are generated, the nonhyperbolic equilibrium point corresponding to the detected Hopf bifurcation displayed a negative solution for X2. Unstable equilibria found using Wolfram Mathematica 9.0: (40, 0, 5, 0, 5) and (34.22, 0.64, 48.24, -0.06, 5.12) when $D^* = 2.03477$.

Results from 5-Dimensional System C

Combination of functions applied:

- Monod model for the growth of Bacteria 1

$$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$$

Results from 5-Dimensional System C

Combination of functions applied:

- Monod model for the growth of Bacteria 1

  $\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$

- Monod model for the growth of Bacteria 2

  $\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2}$

Results from 5-Dimensional System C

Combination of functions applied:

- Monod model for the growth of Bacteria 1

  $\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$

- Monod model for the growth of Bacteria 2

  $\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2}$

- Monod model for the consumptions of Toxin

  $\mu(y) = \frac{m_4 y}{K_4 + y}$

Bifurcations found when Solving 5-Dimensional System C

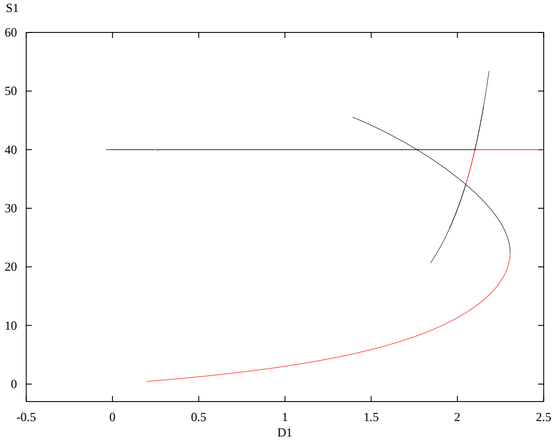


Figure : Illustration of Bifurcations found for 5 Dimensional System C

Results from 5-Dimensional System D

Combination of functions applied:

- Monod model for the growth of Bacteria 1

$$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$$

Results from 5-Dimensional System D

Combination of functions applied:

- Monod model for the growth of Bacteria 1

  $\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$
- Monod model for the growth of Bacteria 2

  $\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2}$

Results from 5-Dimensional System D

Combination of functions applied:

- Monod model for the growth of Bacteria 1

  $$\mu(S_1) = \frac{m_1 S_1}{K_1 + S_1}$$

- Monod model for the growth of Bacteria 2

  $$\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2}$$

- Haldane model for the consumptions of Toxin

  $$\mu(y) = \frac{m_4 y}{K_4 + y + \frac{y^2}{K_{I2}}}$$

Bifurcations found when Solving 5-Dimensional System D

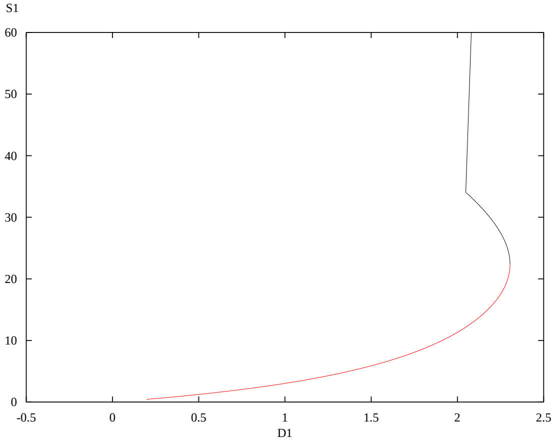


Figure : Illustration of Bifurcations found for 5 Dimensional System D

Parameter values used when solving 6-Dimensional Systems

- $m_1 = 3, \; m_2 = 0.75$

Parameter values used when solving 6-Dimensional Systems

- $m_1 = 3,\ m_2 = 0.75$
- $k_1 = 10,\ k_2 = 5.2254,\ k_3 = 40$

Parameter values used when solving 6-Dimensional Systems

- $m_1 = 3,\ m_2 = 0.75$
- $k_1 = 10,\ k_2 = 5.2254,\ k_3 = 40$
- $K_1 = 0.5,\ K_2 = 0.15,\ K_I = 1$

Parameter values used when solving 6-Dimensional Systems

- $m_1 = 3, \ m_2 = 0.75$
- $k_1 = 10, \ k_2 = 5.2254, \ k_3 = 40$
- $K_1 = 0.5, \ K_2 = 0.15, \ K_I = 1$
- $S_1 = 6, \ S_2 = 0, \ \alpha = 1, \ p = 1, \ r = 0.5.$

6-Dimensional System

Functions applied:

- Haldane model for the growth of Bacteria 2

$$\mu(S_2) = \frac{m_2 S_2}{K_2 + S_2 + \frac{S_2^2}{K_I}}$$

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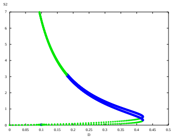
- Periodic inflow of S_{1in} of amplitude "r", in the hopes of generating oscillating solutions.

6-Dimensional System

Bifurcation Results of 6-Dimensional System



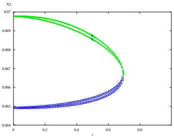


Figure : Saddle-Node Bifurcation of Periodic Orbits, $D^* = 0.4196$ at saddle-node bifurcation point.

Surprising Results: Saddle-Node Bifurcation of Periodic Orbits

6-Dimensional System

Bifurcation Results of 6-Dimensional System

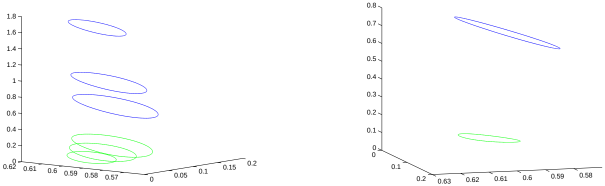
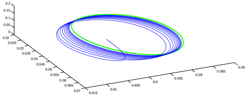


Figure : Periodic Orbits of Opposite Stability converging at Saddle-Node Bifurcation Point. Green: stable, Blue: unstable.

6-Dimensional System

Bifurcation Results of 6-Dimensional System



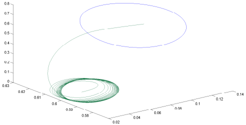


Figure : Stability statuses of Periodic Orbits 6 and 14. Left: Orbit 6, $D^* = 0.2313$ (stable), Right: Orbit 14, $D^* = 0.3937$ (unstable).

Equilibria Results of 6-Dimensional System

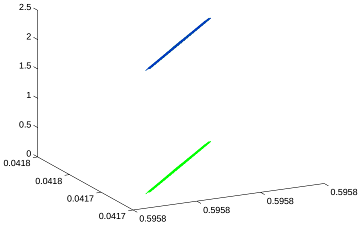


Figure : Unstable Equilibria found for system using specific set of parameter values. (0.0417725, 0.595823, 0.0690168, 0.0761099, 0, 0) and (0.0417725, 0.595823, 2.17338, 0.0235007, 0, 0) when $D^* = 0.2313$.

Further Research to be Done

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- Analysis of environmental conditions to find the most ideal setting for digestion. Provide flexibility for maximum bacteria growth velocity parameters: m_1 , m_2, and m_4.

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