Introduction to Non-Commuting Graphs

G. Hinkle[1] C. Robichaux[2] R. Wood[3]

[1]Department of Mathematics
Rice University

[2]Department of Mathematics
Louisiana State University

[3]Department of Mathematics
Missouri State University

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Outline

Definition of a group

- A group $(G, \star)$ consists of a set G and a binary operation $\star$ that satisfies these four conditions:
  - Closed (For all $x, y \in G$, $x \star y \in G$.)
  - Associative (For all $x, y, z \in G$, $(x \star y) \star z = x \star (y \star z)$.)
  - Identity (There exists a unique element $e \in G$ so that for all $x \in G$, $x \star e = e \star x = x$.)
  - Inverse (For every $x \in G$, there exists a unique $x^{-1} \in G$ so that $x \star x^{-1} = x^{-1} \star x = e$.)

Abelian groups

- An abelian group is a group with the added property of commutativity. That is, for all $x, y \in G$, $x \star y = y \star x$.
- Examples:
  - The integers under addition
  - The integers $\{1, 2, ..., p-1\}$ under multiplication mod p

Non-abelian groups

- A non-abelian group is any group that is not an abelian group. That is, there exist some $x, y \in G$ so that $x \star y \neq y \star x$.
- Examples:
  - Rubik's cube group
  - Dihedral group
- Note that even in a non-abelian group, there are still some pairs of elements that commute with each other.
  - e and any $x \in G$
  - Any $x \in G$ and x^{-1}
  - etc.

Terminology and notation

- The order of a group, written $|G|$, is the number of elements in the group.
- The order of an element $x \in G$, written $o(x)$, is the smallest positive integer n such that $x^n = 1$. All elements of a finite group have finite order.
- The center of a group, written $Z(G)$, is the set of all elements $z \in G$ that commute with all elements of G. If G is abelian, then $Z(G) = G$.
- The centralizer of an element $x \in G$, written $C_G(x)$, is the set of all elements of G that commute with x. If $x \in Z(G)$, then $C_G(x) = G$.
- An AC group is a group G such that for all $x \in (G \setminus Z(G))$, $C_G(x)$ is abelian.

Cyclic group

- The cyclic group of order n is the group generated by one element of order n. That is, it consists of the elements $\{1, a, a^2, a^3, ..., a^{n-1}\}$, with $a^n = 1$.
- Written C_n or $\mathbb{Z}_n$
- $\mathbb{Z}_n = \langle a | a^n = 1 \rangle$
- Abelian

Dihedral group

- The dihedral group of order $2n$ is the group of the symmetries of a regular n-gon. It contains rotations by $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$ of a full rotation as well as each of these rotations followed by a reflection.
- $r =$ rotation by $\frac{1}{n}$
- $s =$ reflection
- $r^n = s^2 = 1$
- Reflecting and then rotating is the same as rotating in the opposite direction and then reflecting, so $sr = r^{-1}s$.
- $D_{2n} = \langle r, s | r^n = s^2 = 1, sr = r^{-1}s \rangle$
- Non-abelian

Symmetric group

- The symmetric group S_n is the group of all permutations of n elements under composition.
- e.g. $\{1,2,3,4,5\} \rightarrow \{4,5,1,3,2\}$
- Cycle notation: $(1,4,3)(2,5)$
- Evaluated from right to left (like functions)
- e.g. $(1,3,2,4)(2,5,3) = (1,3,4)(2,5)$, $(2,5,3)(1,3,2,4) = (1,2,4)(3,5)$
- Non-abelian
- Cayley's theorem: Every group is isomorphic to a subgroup of a symmetric group.

Homomorphisms, isomorphisms, and automorphisms

- A homomorphism from a group $(G, \star)$ to a group $(H, *)$ is a function $\phi : G \to H$ that satisfies $\phi(x \star y) = \phi(x) * \phi(y)$.
- An isomorphism is a bijective homomorphism. If two groups are isomorphic, then they are fundamentally the same, just with different names for the elements.
- An automorphism is an isomorphism from a group to itself.

Direct product

- The direct product of two groups $(G, \star)$ and $(H, *)$ is the group $(G \times H, \bullet)$, where the set $G \times H$ is the Cartesian product of G and H and the operation $\bullet$ acts componentwise:
  - $(g_1, h_1) \bullet (g_2, h_2) = (g_1 \star g_2, h_1 * h_2)$
- Fundamental theorem of finite abelian groups: Every finite abelian group is isomorphic to a direct product of some number of cyclic groups.
- If $\gcd(m, n) = 1$, then $\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn}$.

Semidirect product

- Generalization of the direct product
- Not uniquely defined
- $\phi : H \to Aut(G)$
- $G \rtimes_\phi H$ (or $G \rtimes H$ if the choice of ϕ is clear)
- Any two elements of G interact the same in $G \rtimes H$ as they do in G; any two elements of H interact the same in $G \rtimes H$ as they do in H.
- If $g \in G$ and $h \in H$, then $hgh^{-1} = \phi(h)(g)$. When $\phi(h) = id$, this reduces to the direct product.
- It is sufficient to define how each of the generators of H acts on each of the generators of G.
- $D_{2n} = \mathbb{Z}_n \rtimes_\phi \mathbb{Z}_2$, where $\phi(b)$ is the inverse function in $\mathbb{Z}_n$ (i.e., $\phi(b)(a) = a^{-1}$, so $bab^{-1} = a^{-1}$).

Graph

Definition

Graph: A graph Γ is an ordered pair of disjoint sets (V, E) such that E is a subset of V in the form of unordered pairs. The set V contains all vertices x_i, and the set E contains all edges $x_i x_j$, which connect vertices.

Graph Properties

Definition

Order:

The order of a graph Γ, denoted by $|\Gamma|$ is the number of vertices.

Degree:

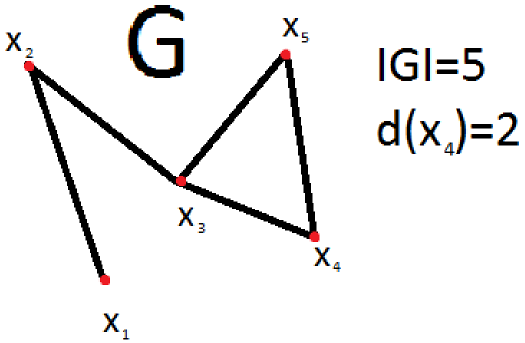
The degree of a vertex, denoted by $d(x)$ is the number of vertices adjacent to a vertex x.

Connected:

A graph is connected provided that there exists a path between each pair of vertices.

Graph Properties

Example



Graph Properties

Definition

Isomorphism: Two graphs are isomorphic if there exists a correspondence between the sets of vertices which preserves adjacency.

Eulerian: A graph is Eulerian if there exists a circuit containing all edges each only once.

Complete:
A graph is complete provided that each pair of vertices has an edge between them or, in other words, are adjacent.

Graph Properties

Definition

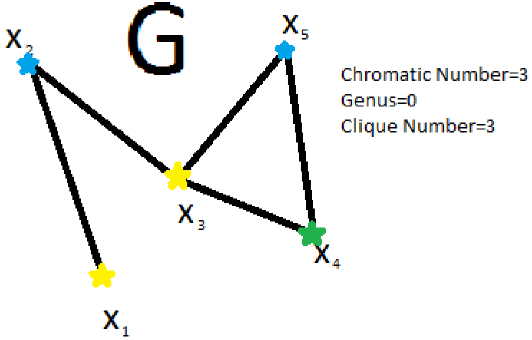
Clique Number:
The clique number of a graph Γ is the maximum order of a complete subgraph of Γ.

Chromatic Number:
The chromatic number of a graph is the minimum number of 'colors' that can be assigned to each vertex such that no vertices of the same color are adjacent.

Genus:
The genus of a graph is the minimum number of handles that must be added to a surface such that the graph may be drawn on the surface with no edges crossing.

Graph Properties

Example

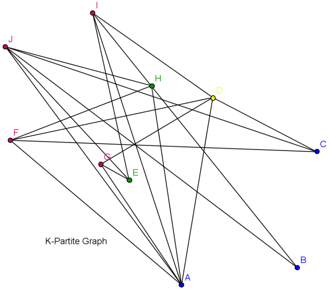


Graph Types

Definition

K-partite Graph:

A graph is k-partite if the vertices can be separated in classes $V_1, V_2, ..., V_k$ such that $V = V_1 \cup V_2 \cup ... \cup V_k$, $V_i \cap V_j = \emptyset$ for $1 \leq i < j \leq k$, and no edge joins two vertices of the same class.



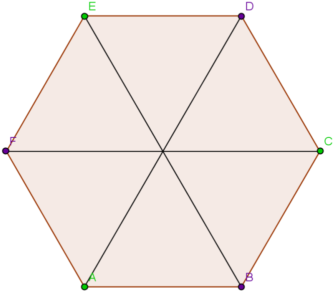
K-Partite Graph

Graph Types

Definition

Complete K-Partite Graph:

A graph is complete k-partite denoted by $K_{n_1,\ldots,n_k}$ if the graph has every n_i vertices in the ith class and contains all edges joining vertices in distinct classes.



Complete K-Partite Graph ($K_{3,3}$)

Outline

Non-Commuting Graph

Definition

- **Definition**

  A non-commuting graph of a group G is the vertex set $G - Z(G)$ where two distinct vertices x and y are joined by an edge whenever $xy \neq yx$ is called the non-commuting graph of a group.

  - The non-commuting graph of a group G will be denoted $\Gamma(G)$.
  - First considered by Paul Erdos in 1975

Example

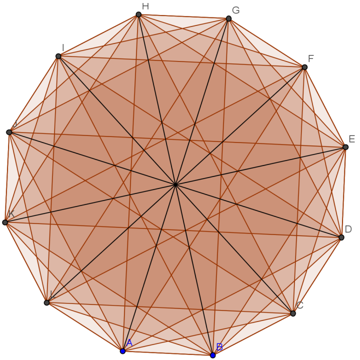


Figure : $\Gamma(M_{16})$

Example

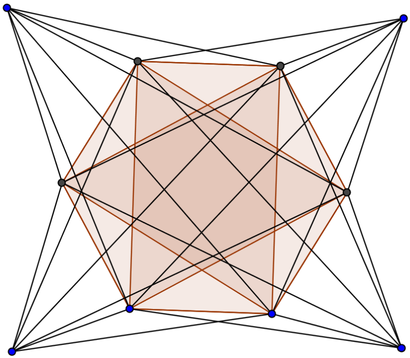


Figure : $\Gamma(D_{12})$

Example

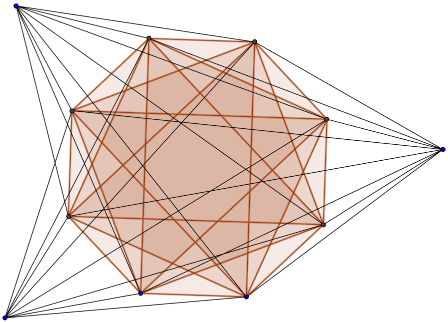


Figure : $\Gamma(A_4)$

Outline

Characteristics of the Graph Related to the Group

- Genus
- Characteristic Polynomial
- Chromatic Number
- Cop Number
- Graph Isomophisms
- Eulerian
- Cliche Number

Summary

- Our research is looking at the non-commuting graphs of non-abelian groups and their properties.
- Understanding the non-commuting graph of a group helps us understand the structure of the group.
- Presentations:
  - R. Wood: Graph Genus and Other Properties
  - C. Robichaux: Characteristic Polynomial
  - G. Hinkle: Eulerian Non-commuting Graphs / Automorphism Groups