Dynamics of a Networked Connectivity Model of Waterborne Disease Epidemics

A. Edwards, D. Mercadante, C. Retamoza

REU Final Presentation
July 31, 2014

Overview

- Background on waterborne diseases
- Introduction to epidemic model and parameters
- Local stability analysis
- Lyapunov functions and global stability analysis
- Bifurcation analysis
- (Non)Existence of periodic solutions
- Our contribution

Waterborne Diseases

- Mainly caused by protozoa or bacteria present in water.
- Waterborne diseases are one of the leading causes of death in low-income countries, particularly affecting infants and children in those areas.
- The models we have been studying use mainly V. cholera as an example.
- Vast majority of models in the literature consider one single community.
- First networked connectivity model introduced in late 2012.

Why Are We Studying This Model?

- Learn about dynamical systems
- To gain a better understanding of:
  - Onset conditions for outbreak
  - The spread of diseases by hydrological means

- Insight
- Emergency management
- New health-care resources

Reproduction Matrices/Values

- R_0 is known as the basic reproduction number in SIR and related models.
    - This stands only when $n = 1$ (most research articles study only this case)
    - Outbreak occurs when $R_0 > 1$
    - With networked connectivity, this statement does not hold.

- For connectivity models, we study G_0, or the generalized reproduction matrix. When the dominant eigenvalue of this matrix crosses the value one, an outbreak will occur. This value is independent from the values of R_{0i}.

Bifurcations

- Bifurcation occurs when small changes in parameters imply drastic changes in the solutions, number of equilibrium points, or their stability properties.
- Disease-free equilibrium
- Transcritical Bifurcation

The Model

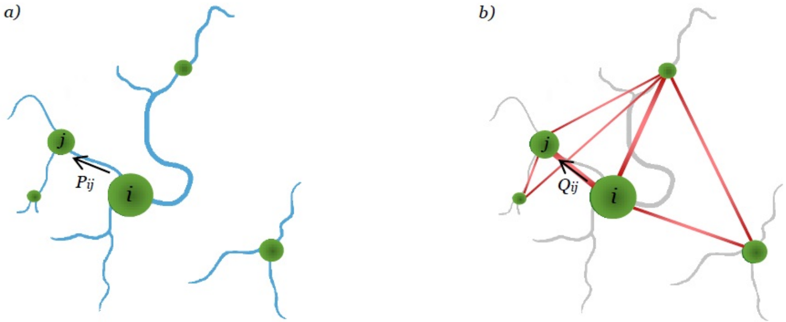
Consider n communities (nodes). For $i = 1, \ldots, n$, let

- S_i = number of susceptibles in node i
- I_i = number of infectives in node i
- B_i = concentration of bacteria in water at node i

- Spatially explicit nonlinear differential model

- Communities are connected by hydrological and human mobility networks through which a disease can spread.

Strongly Connected



Graph $\Gamma(\mathbf{P} \cup \mathbf{Q})$ is strongly connected.

The Epidemic Model (Gatto et al.)

For $i = 1, \ldots, n$ (3n differential equations)

$$
\begin{aligned}
\frac{dS_i}{dt} &= \mu(H_i - S_i) - \left[(1-m_s)\beta_i f(B_i) + m_s \sum_{j=1}^{n} Q_{ij}\beta_j f(B_j)\right] S_i \\
\frac{dI_i}{dt} &= \left[(1-m_s)\beta_i f(B_i) + m_s \sum_{j=1}^{n} Q_{ij}\beta_j f(B_j)\right] S_i - \phi I_i \\
\frac{dB_i}{dt} &= -\mu_B B_i + l\left(\sum_{j=1}^{n} P_{ji}\frac{W_j}{W_i}B_j - B_i\right) + \frac{p_i}{W_i}\left[(1-m_I)I_i + \sum_{j=1}^{n} m_I Q_{ji} I_j\right]
\end{aligned}
$$

where $f(B_i) = \dfrac{B_i}{K + B_i}$.

(Local)Stability Analysis of a Single Community

For $n = 1$, the model is

$$\begin{aligned}
\frac{dS}{dt} &= \mu(H - S) - \beta f(B)S \\
\frac{dI}{dt} &= \beta f(B)S - \phi I \\
\frac{dB}{dt} &= (nb - mb)B + pI
\end{aligned}$$

where $f(B) = \dfrac{B}{K + B}$.

Linearization

Definition

A set $S \subset R^n$ is said to be **invariant** with respect to $x' = f(x)$ if $x(0) \in S \implies x(t) \in S$ for all $t \geq 0$.

Linearization Technique:

Consider the nonlinear system $x' = f(x)$ in R^n

Let x_0 be a hyperbolic equilibrium point $(f(x_0) = 0)$

Local stability analysis: Study the linear system $x' = Ax$, where

$$A = Df(x_0), \qquad Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

(Local)Stability Analysis of a Single Community

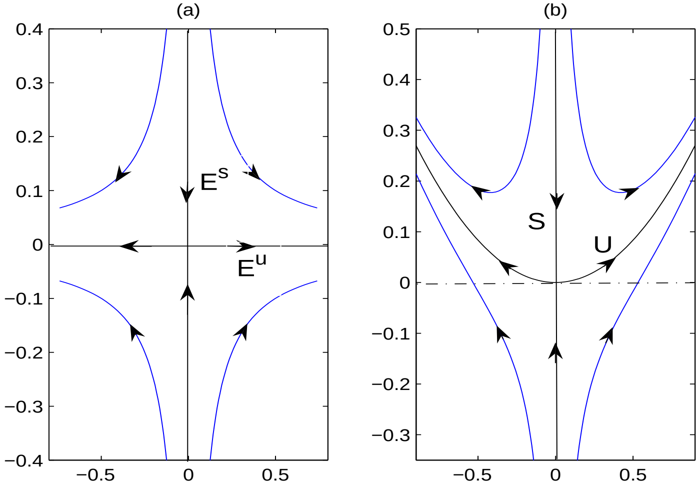
Theorem

Stable Manifold Theorem: *Let E be an open subset of R^n containing the equilibrium point x_0 of $x' = f(x)$, and let $f \in C^1(E)$. Suppose that $Df(x_0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold* **S** *tangent to the stable subspace E^s of the linear system $x' = \mathbf{A}x$ at x_0 such that* **S** *is invariant, and solutions approach x_0 as $t \to \infty$. And there exists an $n-k$ dimensional differentiable manifold* **U** *tangent to the unstable subspace E^u of $x' = \mathbf{A}x$ at x_0 such that* **U** *is invariant and solutions move away from x_0 as $t \to \infty$.*

Theorem

Hartman - Grobman Theorem: *Let E be and open subset of R^n containing a hyperbolic equilibrium point x_0 of $x' = f(x)$, and let $f \in C^1(E)$. Then there exists a homeomorphism* **H** *of an open set U containing x_0 into an open set V containing x_0 such that H maps trajectories of $x' = f(x)$ near x_0 onto the trajectories of $x' = \mathbf{A}x$ near x_0.*

(Local)Stability Analysis of a Single Community



(Local)Stability Analysis of a Single Community

For this model, the general Jacobian is

$$\mathbf{J}(S, I, B) = \begin{bmatrix} -\mu - \dfrac{\beta B}{K+B} & 0 & -\dfrac{\beta S K}{(K+B)^2} \\ \dfrac{\beta B}{K+B} & -\phi & \dfrac{\beta S K}{(K+B)^2} \\ 0 & p & nb - mb \end{bmatrix}$$

Disease-Free Equilibrium

$$S_1^* = H,\ I_1^* = 0,\ B_1^* = 0$$

$$A = \mathbf{J}(S_1^*, I_1^*, B_1^*) = \begin{bmatrix} -\mu & 0 & -\dfrac{\beta H}{K} \\ 0 & -\phi & \dfrac{\beta H}{K} \\ 0 & p & nb - mb \end{bmatrix}$$

Characteristic Equation:

$$\begin{aligned} P(\lambda) &= \lambda^3 + (\mu + \phi - nb + mb)\lambda^2 + \\ &+ \left(\mu\phi + (\mu + \phi)(mb - nb) - \frac{\beta H p}{K}\right)\lambda + \\ &+ \mu\phi mb - \mu\phi nb - \frac{\mu\beta H p}{K} \end{aligned}$$

Routh-Hurwitz Criteria

Theorem (Routh-Hurwitz)

Given the polynomial,

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n,$$

where the coefficients a_i *are real constants, i=1,...,n. For polynomials of degree n=3, the Routh-Hurwitz criteria are summarized by*

$$a_1 > 0, \;\; a_3 > 0, \quad \textit{and} \quad a_1 a_2 > a_3.$$

These are necessary and sufficient conditions for all of the roots of the characteristic polynomial (with real coefficients) to lie in the left half of the complex plane (implying stability).

Disease-Free Equilibrium

By applying the $Routh - Hurwitz\ Criteria$ to the characteristic equation, it was found that $a_1 > 0$, $a_3 > 0$, and $a_1 a_2 > a_3$, were true for the following inequalities,

$$mb > nb, \;\; S_c = \frac{\phi K(mb - nb)}{\beta p} > H = S_0.$$

Let $R_0 = \frac{S_0}{S_c}$, then $R_0 < 1$ and our equilibrium is stable.

Basic Reproduction Number R_0

In the previous model, $R_0 = \dfrac{\beta p S_0}{\phi K(mb - nb)} = \dfrac{S_0}{S_c}$. How is R_0 determined in general?

Theorem (Castillo-Chavez et al.)

Consider the $(n_1 + n_2 + n_3)$-dimensional system

$$\begin{aligned} x' &= f(x, E, I) \\ E' &= g(x, E, I) \\ I' &= h(x, E, I). \end{aligned}$$

Let $(x^*, 0, 0)$ be the disease-free equilibrium. Assume the equation $g(x^*, E, I) = 0$ implicitly determines a function $E = \tilde{g}(x^*, I)$, and let $A = D_I h(x^*, \tilde{g}(x^*, 0), 0)$. Assume further that A can be written as $A = M - D$, where $M \geq 0$ and $D > 0$ is diagonal. Then,

$$R_0 = \rho(MD^{-1}).$$

Lyapunov Functions and Global Stability

Definition

Let E be an open set in R^n, and let $x_0 \in E$. A function $Q : E \subset R^n \to R$ is called a **Lyapunov function** for $x' = f(x)$ if $Q \in C^1(E)$ and satisfies $Q(x_0) = 0$, and $Q(x) > 0$ if $x \neq x_0$.

Theorem

Let $R_0 \leq 1, mb > nb$, and let $\mathbf{w}$ be a left eigenvector of the matrix $\mathbf{V}^{-1}\mathbf{F}$ corresponding to $R_0 = \rho(\mathbf{V}^{-1}\mathbf{F})$, then the function $Q(\mathbf{x}) = \mathbf{w}^T\mathbf{V}^{-1}\mathbf{x}$ is a Lyapunov function of the system of equations for n=1 satisfying $Q'(\mathbf{x}(t)) \leq 0$.

Lyapunov Functions and Global Stability

Our system can be written as compartmental model

$$\mathbf{x}' = \mathbf{F}^*(x,y) - \mathbf{V}^*(x,y), \qquad\qquad y' = g(x,y)$$

where $x = [I, B]^T \in R^2$ is the disease compartment and $y = S \in R$ is the disease-free compartment, respectively. And

$$\mathbf{F} = \left[\frac{\partial F_i^*}{\partial x_j}(0, y_0) \right] \quad \text{and} \quad \mathbf{V} = \left[\frac{\partial V_i^*}{\partial x_j}(0, y_0) \right], \qquad 1 \le i, j \le n$$

Lyapunov Functions and Global Stability

Let $\mathbf{F}^* = \left[\frac{\beta SB}{K+B} \quad p\,I\right]^T$ and $\mathbf{V}^* = [-\phi I \quad (mb-nb)B]^T$, then the matrices $\mathbf{F}$ and $\mathbf{V}$ are given by

$$\mathbf{F} = \begin{bmatrix} 0 & \frac{BH}{K} \\ p & 0 \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} \phi & 0 \\ 0 & mb-nb \end{bmatrix}.$$

The matrix

$$\mathbf{V}^{-1}\mathbf{F} = \begin{bmatrix} 0 & \frac{\beta H}{K\phi} \\ \frac{p}{mb-nb} & 0 \end{bmatrix}$$

leads us to

$$\rho(\mathbf{V}^{-1}\mathbf{F}) = R_0 = \sqrt{\frac{\beta Hp}{K\phi(mb-nb)}}$$

Lyapunov Functions and Global Stability

The left eigenvector is found to be $\mathbf{w}^T = \begin{bmatrix} 1 & R_0(\frac{mb-nb}{p}) \end{bmatrix}$. As $Q(\mathbf{x}) = \mathbf{w}^T\mathbf{V}^{-1}\mathbf{x}$, we can now write it as

$$Q = \begin{bmatrix} 1 & R_0(\frac{mb-nb}{p}) \end{bmatrix} \begin{bmatrix} \frac{1}{\phi} & 0 \\ 0 & \frac{1}{mb-nb} \end{bmatrix} \begin{bmatrix} I \\ B \end{bmatrix} = \frac{I}{\phi} + \frac{R_0 B}{p}.$$

Now let $\mathbf{f}(x,y) = (\mathbf{F}-\mathbf{V})x - \mathbf{x}'$ and $Q' = \mathbf{w}^T\mathbf{V}^{-1}\mathbf{x}' = (R_0 - 1)\mathbf{w}^T\mathbf{x} - \mathbf{w}^T\mathbf{V}^{-1}\mathbf{f}(x,y)$, where

$$\mathbf{f}(x,y) = \begin{bmatrix} \frac{\beta HB}{k} - \frac{\beta SB}{K+B} \\ 0 \end{bmatrix}.$$

$$Q' = (R_0 - 1)\left(I + \frac{R_0 B}{p}(mb-nb)\right) - \frac{\beta B}{\phi}\left(\frac{H}{K} - \frac{S}{K+B}\right) \leq 0.$$

Lyapunov Functions and Global Stability

Theorem (LaSalle's Invariance Principle)

Let $\Omega \subset D \subset R^n$ be a compact invariant set with respect to $x' = f(x)$. Let $Q : D \to R$ be a C^1 function such that $Q'(x(t)) \leq 0$ in Ω. Let $E \subset \Omega$ be the set of all points in Ω where $Q'(x) = 0$. Let $M \subset E$ be the largest invariant set in E. Then

$$\lim_{t\to\infty}\left[\inf_{y\in M}\|x(t)-y\|\right] = 0.$$

That is, every solution starting in Ω approaches M as $t \to \infty$.

Theorem

Let Ω be any compact invariant set containing the disease-free equilibrium point. Let $\mathbf{f}$, $\mathbf{F}$, and $\mathbf{V}$ be defined as above. Suppose $R_0 < 1$, $mb > nb$, and $\mathbf{f}(x, y) \geq 0$, $\mathbf{F} \geq 0$, $\mathbf{V}^{-1} \geq 0$, $\mathbf{f}(0, y) = 0$. Also, assume the disease-free system $y' = g(0, y)$ has a unique equilibrium $y = y_0 > 0$ that is globally asymptotically stable in R. Then, the disease-free equilibrium is globally asymptotically stable in Ω.

Lyapunov Functions and Global Stability

The disease-free system $y' = g(0, y)$ is equivalent to $S' = \mu H - \mu S$, whose solution is $S = H + e^{-\mu t} C$.

$y_0 = H$ is globally asymptotically stable in the disease-free system.

$Q' = (R_0 - 1)\mathbf{w}^T\mathbf{x} - \mathbf{w}^T\mathbf{V}^{-1}\mathbf{f}(x, y)$. Assuming $Q' = 0$ implies $\mathbf{x} = 0$. Then, the set of all points where $Q' = 0$ is $E = \{(I, B, S) : I = B = 0\}$. The largest and only invariant set in E is $(0, 0, H)$.

Therefore, the disease-free equilibrium is globally asymptotically stable in Ω.

Endemic Equilibrium

The endemic equilibrium represents a single isolated community where there is interaction between susceptibles, infectives, and bacteria in water.

$$S_2^* = \frac{\mu H(K+B)}{\mu(K+B)+\beta B}$$

$$I_2^* = \frac{\beta B \mu H}{\phi(\mu(K+B)+\beta B)}$$

$$B_2^* = \frac{\mu(p\beta H + K\phi(nb-mb))}{\phi(\mu+\beta)(mb-nb)}$$

Endemic Equilibrium

Let $A = \dfrac{\beta B}{K+B}$ and $C = \dfrac{\beta S K}{(K+B)^2}$,

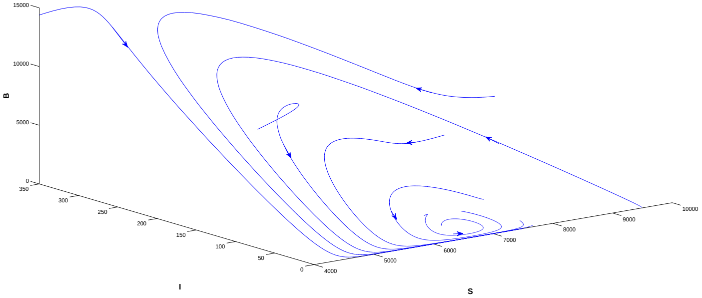
$$\mathbf{J}(S_2^*, I_2^*, B_2^*) = \begin{bmatrix} -(\mu + A) & 0 & -C \\ A & -\phi & C \\ 0 & p & nb - mb \end{bmatrix}$$

Using Routh-Hurwitz criteria: As long as $H > S_c$ (or $R_0 > 1$), all real parts of the eigenvalues are negative, making the endemic equilibrium locally stable.

Therefore, endemic equilibrium is stable exactly when disease-free equilibrium is unstable.

Phase Portrait

$\beta = 1,\ \phi = 0.2,\ \mu = 0.0001,\ p = 10,\ mb = 0.4,\ nb = 0.067,\ H = 10000,\ K = 1000000$ and $S_c = 6660$



Bifurcations

Theorem (Sotomayor)

Consider $x' = \mathbf{f}(\mathbf{x}, \alpha)$, *where* $\mathbf{x} \in R^n$ *and* α *is a parameter. Let* $(\mathbf{x}_0, \alpha_0)$ *be an equilibrium point and assume* $\mathbf{A} = D\mathbf{f}(\mathbf{x}_0, \alpha_0)$ *has a simple eigenvalue* $\lambda = 0$ *with eigenvector* $\mathbf{v}$, *and left eigenvector* $\mathbf{w}$. *If*

$$\mathbf{w}^T \mathbf{f}_\alpha(x_0, \alpha_0) = 0,$$

$$\mathbf{w}^T [D\mathbf{f}_\alpha(\mathbf{x}_0, \boldsymbol{\alpha}_0)\mathbf{v}] \neq 0,$$

$$\mathbf{w}^T [D^2\mathbf{f}(\mathbf{x}_0, \boldsymbol{\alpha}_0)(\mathbf{v}, \mathbf{v})] \neq 0$$

then a transcritical bifurcation occurs at $(\mathbf{x}_0, \boldsymbol{\alpha}_0)$.

Note: $D^2\mathbf{f}(\mathbf{x}_0)(\mathbf{u}, \mathbf{v}) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \frac{\partial^2 \mathbf{f}(\mathbf{x}_0)}{\partial x_{j_1} \partial x_{j_2}} u_{j_1} v_{j_2}$

Bifurcations

Theorem

The $n = 1$ system undergoes a transcritical bifurcation at the disease-free equilibrium point $(S_1^*, I_1^*, B_1^*) = (H, 0, 0)$ when $p = \dfrac{\phi(mb - nb)K}{\beta H}$, and $mb > nb$.

Proof:

Stability of the system at the equilibrium point depends on the bottom right 2×2 matrix of $\mathbf{J}$, given by

$$\mathbf{J} = \begin{bmatrix} -\mu & 0 & -\frac{\beta H}{K} \\ 0 & -\phi & \frac{\beta H}{K} \\ 0 & p & nb - mb \end{bmatrix} \quad \mathbf{J^*} = \begin{bmatrix} -\phi & \frac{\beta H}{K} \\ p & nb - mb \end{bmatrix}.$$

When the $\det \mathbf{J^*} = 0$, $p = p_0 = \dfrac{\phi(mb - nb)K}{\beta H}$.

Sotomayor's Theorem

To satisfy the first condition of the theorem, $\mathbf{w}^T \mathbf{f}_p(\mathbf{x}_0, p_0) = 0$, we have $\mathbf{w}^T = \begin{bmatrix} 0 & 1 & \frac{\phi}{p} \end{bmatrix}$, and $\mathbf{f}_p = \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}$. Then,

$$\mathbf{w}^T \mathbf{f}_p(\mathbf{x}_0, p_0) = \begin{bmatrix} 0 & 1 & \frac{\phi}{p} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

Sotomayor's Theorem

To satisfy the second condition of the theorem,

$\mathbf{w}^T[D\mathbf{f}_p(\mathbf{x}_0, \mathbf{p}_0)\mathbf{v}] \neq 0$, we have $D\mathbf{f}_p = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$\mathbf{w}^T[D\mathbf{f}_p(\mathbf{x}_0, p_0)\mathbf{v}] = \begin{bmatrix} 0 & 1 & \frac{\phi}{p} \end{bmatrix} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{-\beta H}{K\mu} \\ \frac{mb - nb}{p} \\ 1 \end{bmatrix} \right)$$

$$= \frac{\phi(nb - mb)}{p^2} \neq 0.$$

Sotomayor's Theorem

To satisfy the third condition of the theorem, $\mathbf{w}^T[D^2\mathbf{f}(\mathbf{x}_0, \mathbf{p}_0)(\mathbf{v}, \mathbf{v})] \neq 0$, we have

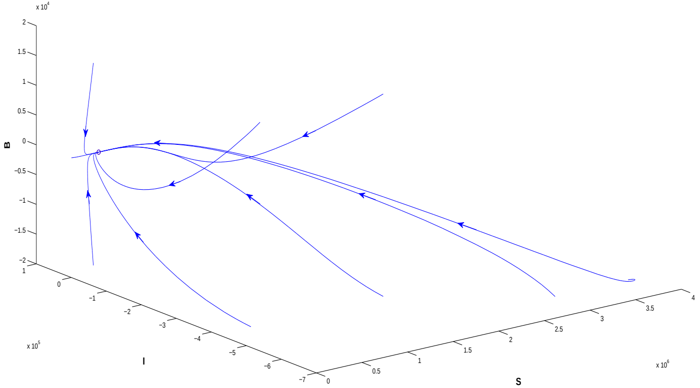
$$D^2\mathbf{f}(\mathbf{x}_0, p_0)(\mathbf{v}, \mathbf{v}) = \begin{bmatrix} \frac{2\beta H}{K^2}(\frac{\beta}{\mu} + 1) \\ -\frac{2\beta H}{K^2}(\frac{\beta}{\mu} + 1) \\ 0 \end{bmatrix}. \quad \text{Then}$$

$$\mathbf{w}^T[D^2\mathbf{f}(\mathbf{x}_0, p_0)(\mathbf{v}, \mathbf{v})] = \begin{bmatrix} 0 & 1 & \frac{\phi}{p} \end{bmatrix} \begin{bmatrix} \frac{2\beta H}{K^2}(\frac{\beta}{\mu} + 1) \\ -\frac{2\beta H}{K^2}(\frac{\beta}{\mu} + 1) \\ 0 \end{bmatrix}$$

$$= -\frac{2\beta H}{K^2}(\frac{\beta}{\mu} + 1) \neq 0.$$

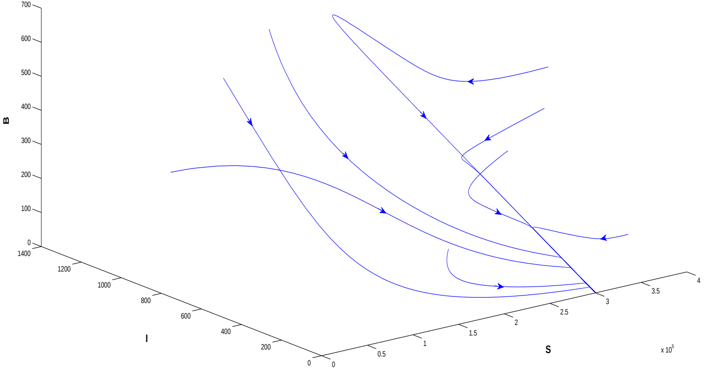
Bifurcations

Figure : $p < p_0$



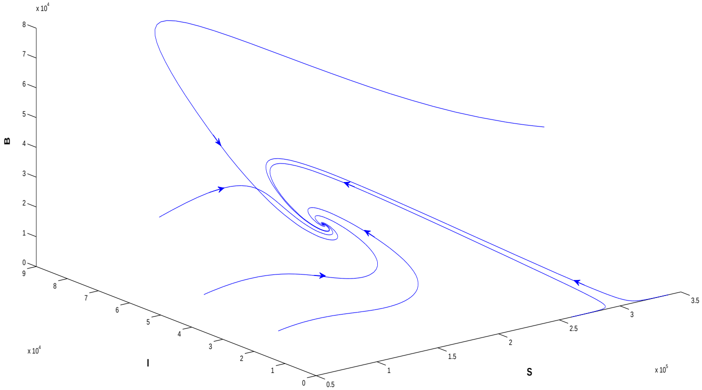
Bifurcations

Figure : $p = p_0$



Bifurcations

Figure : $p > p_0$



Hopf Bifurcation

A stable focus (solutions spiral toward the equilibrium) corresponds to $\lambda = a \pm bi$ where $a < 0$. When $a = 0$, stability may be lost, periodic orbits may appear, and a Hopf Bifurcation could occur. The main the condition to prove the existence of a Hopf Bifurcation is $\lambda = \pm bi$.

Proposition: A Hopf Bifurcation at the DFE does not exist.

Proof.

For the lower right block of Jacobian at the equilibrium, we have: $T = -\phi + (nb - mb)$ and $D = (-\phi(nb - mb)) - (p\beta H/K)$. To obtain purely imaginary eigenvalues, we must have $T = 0$ and $D > 0$. But $T = 0$ is only possible if $nb > mb$. But then $D < 0$. □

Periodic Solutions

Theorem (Bendixon's Criterion in R^n)

A simple closed rectifiable curve which is invariant with respect to the differential equation, $\frac{dx}{dt} = f(x)$*, cannot exist if any one of the following conditions is satisfied on* R^n*:*

(i) $\sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right) : 1 \leq r < s \leq n\right\} < 0,$

(ii) $\sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right) : 1 \leq r < s \leq n\right\} < 0,$

(iii) $\lambda_1 + \lambda_2 < 0,$

iv) $\inf\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial f_q}{\partial x_r}\right| + \left|\frac{\partial f_q}{\partial x_s}\right|\right) : 1 \leq r < s \leq n\right\} > 0,$

(v) $\sup\left\{\frac{\partial f_r}{\partial x_r} + \frac{\partial f_s}{\partial x_s} + \sum_{q\neq r,s}\left(\left|\frac{\partial f_r}{\partial x_q}\right| + \left|\frac{\partial f_s}{\partial x_q}\right|\right) : 1 \leq r < s \leq n\right\} > 0,$

(vi) $\lambda_{n-1} + \lambda_n > 0.$

where the λ_i *are the ordered eigenvalues of* $M = \frac{1}{2}\left[Df(x) + Df(x)^T\right]$.

Periodic Solutions

$$\sup\left\{-\mu-\beta(\tfrac{B}{K+B})-\phi+p,\ -\phi+(nb-mb)+\tfrac{\beta SK}{(K+B)^2},\ -\mu+(nb-mb)+\tfrac{\beta SK}{(K+B)^2}\right\}<0$$

$$\begin{aligned}
p &\leq \mu+\beta(\tfrac{B}{K+B})+\phi \\
p &\leq (nb-mb)+\tfrac{\beta SK}{(K+B)^2}+\mu+\beta(\tfrac{B}{K+B}) \\
p &< (nb-mb)+\tfrac{\beta SK}{(K+B)^2}+\phi+\beta(\tfrac{B}{K+B})
\end{aligned}$$

$$\begin{aligned}
\mu &\geq -nb+mb+-\tfrac{\beta SK}{(K+B)^2}-\tfrac{\beta B}{K+B}+p \\
\phi &\leq \mu \\
\phi &> nb-mb+\tfrac{\beta SK}{(K+B)^2}
\end{aligned}$$

$$\begin{aligned}
\phi &\geq -nb+mb+-\tfrac{\beta SK}{(K+B)^2}-\tfrac{\beta B}{K+B}+p \\
\phi &\geq \mu \\
\phi &> nb-mb+\tfrac{\beta SK}{(K+B)^2}
\end{aligned}$$

The Networked Epidemic Model

First consider $n = 3 \;\; (i = 1, 2, 3)$. This gives 9 differential equations:

$$
\begin{aligned}
\frac{dS_i}{dt} &= \mu(H_i - S_i) - \left[(1 - m_s)\beta_i f(B_i) + m_s \sum_{j=1}^{n} Q_{ij}\beta_j f(B_j)\right] S_i \\
\frac{dI_i}{dt} &= \left[(1 - m_s)\beta_i f(B_i) + m_s \sum_{j=1}^{n} Q_{ij}\beta_j f(B_j)\right] S_i - \phi I_i \\
\frac{dB_i}{dt} &= -\mu_B B_i + l\left(\sum_{j=1}^{n} P_{ji}\frac{W_j}{W_i} B_j - B_i\right) + \frac{p_i}{W_i}\left[(1 - m_I)I_i + \sum_{j=1}^{n} m_I Q_{ji} I_j\right]
\end{aligned}
$$

(Local) Stability Analysis of Three Communities

Jacobian **J** at disease-free equilibrium is

$$\begin{bmatrix}
-\mu & 0 & 0 & 0 & 0 & 0 & -H_1(1-m_S)\beta_1 & -H_2 m_S Q_{12}\beta_2 & -H_3 m_S Q_{13}\beta_3 \\
0 & -\mu & 0 & 0 & 0 & 0 & -H_1 m_S Q_{21}\beta_1 & -H_2(1-m_S)\beta_2 & -H_3 m_S Q_{23}\beta_3 \\
0 & 0 & -\mu & 0 & 0 & 0 & -H_1 m_S Q_{31}\beta_1 & -H_2 m_S Q_{32}\beta_2 & -H_3(1-m_S)\beta_3 \\
0 & 0 & 0 & -\phi & 0 & 0 & H_1(1-m_S)\beta_1 & H_2 m_S Q_{12}\beta_2 & H_3 m_S Q_{13}\beta_3 \\
0 & 0 & 0 & 0 & -\phi & 0 & H_1 m_S Q_{21}\beta_1 & H_2(1-m_S)\beta_2 & H_3 m_S Q_{23}\beta_3 \\
0 & 0 & 0 & 0 & 0 & -\phi & H_1 m_S Q_{31}\beta_1 & H_2 m_S Q_{32}\beta_2 & H_3(1-m_S)\beta_3 \\
0 & 0 & 0 & \frac{p_1(1-m_I)}{W_1 K} & \frac{p_1 m_I Q_{21}}{W_1 K} & \frac{p_1 m_I Q_{31}}{W_1 K} & -\mu_B - l & lP_{21}\frac{W_2}{W_1} & lP_{31}\frac{W_3}{W_1} \\
0 & 0 & 0 & \frac{p_2 m_I Q_{12}}{W_2 K} & \frac{p_2(1-m_I)}{W_2 K} & \frac{p_2 m_I Q_{32}}{W_2 K} & lP_{12}\frac{W_1}{W_2} & -\mu_B - l & lP_{32}\frac{W_3}{W_2} \\
0 & 0 & 0 & \frac{p_3 m_I Q_{13}}{W_3 K} & \frac{p_3 m_I Q_{23}}{W_3 K} & \frac{p_3(1-m_I)}{W_3 K} & lP_{13}\frac{W_1}{W_3} & lP_{23}\frac{W_2}{W_3} & -\mu_B - l
\end{bmatrix}$$

Note: $B_i = B_i/K$

(Local) Stability Analysis of Three Communities

We need to find general conditions for (local) stability.

We can block the Jacobian $\mathbf{J}$ as

$$\mathbf{J} = \begin{bmatrix} \mathbf{J}_{11} & 0 & \mathbf{J}_{13} \\ 0 & \mathbf{J}_{22} & \mathbf{J}_{23} \\ 0 & \mathbf{J}_{32} & \mathbf{J}_{33} \end{bmatrix} \qquad \lambda(\mathbf{J}) = \lambda(\mathbf{J}_{11}) \cup \lambda \begin{bmatrix} \mathbf{J}_{22} & \mathbf{J}_{23} \\ \mathbf{J}_{32} & \mathbf{J}_{33} \end{bmatrix}$$

$$\mathbf{J}^* = \begin{bmatrix} \mathbf{J}_{22} & \mathbf{J}_{23} \\ \mathbf{J}_{32} & \mathbf{J}_{33} \end{bmatrix}$$

Each block $\mathbf{J}_{ij}$ is a 3×3 matrix, and

$\mathbf{J}_{11} = \mathrm{diag}(-\mu)$.

Irreducibility

Definition

A matrix A_{nxn} is reducible if it is similar to a block triangular matrix: There is a permutation matrix P such that

$$P^T AP = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, 0 \in \mathbf{R}^{r \times (n-r)}, (1 \leq r \leq n-1)$$

A matrix is irreducible if it is not reducible.

(Local) Stability Analysis of Three Communities

J* is given by

$$\begin{bmatrix} -\phi & 0 & 0 & H_1(1-m_S)\beta_1 & H_2 m_S Q_{12}\beta_2 & H_3 m_S Q_{13}\beta_3 \\ 0 & -\phi & 0 & H_1 m_S Q_{21}\beta_1 & H_2(1-m_S)\beta_2 & H_3 m_S Q_{23}\beta_3 \\ 0 & 0 & -\phi & H_1 m_S Q_{31}\beta_1 & H_2 m_S Q_{32}\beta_2 & H_3(1-m_S)\beta_3 \\ \frac{p_1(1-m_I)}{W_1 K} & \frac{p_1 m_I Q_{21}}{W_1 K} & \frac{p_1 m_I Q_{31}}{W_1 K} & -\mu_B - I & IP_{21}\frac{W_2}{W_1} & IP_{31}\frac{W_3}{W_1} \\ \frac{p_2 m_I Q_{12}}{W_2 K} & \frac{p_2(1-m_I)}{W_2 K} & \frac{p_2 m_I Q_{32}}{W_2 K} & IP_{12}\frac{W_1}{W_2} & -\mu_B - I & IP_{32}\frac{W_3}{W_2} \\ \frac{p_3 m_I Q_{13}}{W_3 K} & \frac{p_3 m_I Q_{23}}{W_3 K} & \frac{p_3(1-m_I)}{W_3 K} & IP_{13}\frac{W_1}{W_3} & IP_{23}\frac{W_2}{W_3} & -\mu_B - I \end{bmatrix}$$

J* is irreducible.

Metzler Matrix

Definition

A matrix $\mathbf{M}$ is said to be Metzler if all of the off-diagonal entries are nonnegative.

1. If you have an $\mathbf{M}_{nxn}$ Metzler matrix, then the eigenvalue with maximum real part is real.

$$\alpha(\mathbf{M}) = \max_{i} Re\lambda_i(\mathbf{M}), \quad i = 1, 2, \ldots, n$$

2. The Metzler matrix, $\mathbf{M}$, is asymptotically stable iff $\alpha(\mathbf{M}) < 0$

Therefore $\mathbf{J}^*$ is a Metzler matrix.

(Local) Stability Analysis of Three Communities

A very useful variation of Perron-Frobenius theorem:

Theorem

Let $\mathbf{M} \in \mathbb{R}^{nxn}$ *be a Metzler matrix. If* $\mathbf{M}$ *is irreducible, then* $\lambda^* = \alpha(M)$ *is a simple eigenvalue, and its associated eigenvector is positive.*

When λ^* reaches zero, the $\det(\mathbf{J^*}) = 0$ and the disease free equilibrium point will lose stability.

Determinant of $\mathbf{J}^*$

$$\mathbf{J}^* = \begin{bmatrix} \mathbf{J}_{22} & \mathbf{J}_{23} \\ \mathbf{J}_{32} & \mathbf{J}_{33} \end{bmatrix}, \qquad \det(\mathbf{J}^*) = (\mathbf{J}_{22}\mathbf{J}_{33}) - (\mathbf{J}_{23}\mathbf{J}_{32}) = 0$$

$$\mathbf{J}_{22} = -\phi\mathbf{U_3}$$
$$\mathbf{J}_{23} = m_S\mathbf{HQ}\boldsymbol{\beta} + (1-m_S)\mathbf{H}\boldsymbol{\beta}$$
$$\mathbf{J}_{23} = \frac{m_I}{K}\mathbf{pW^{-1}Q^T} + \frac{1-m_I}{K}\mathbf{pW^{-1}}$$
$$\mathbf{J}_{33} = -(\mu_B + l)\mathbf{U_3} + l\,\mathbf{W^{-1}P^TW}$$

$$\begin{aligned}\det(\mathbf{J}^*) \quad = \quad & \det\Big[\phi(\mu_B + l)\mathbf{U_3} - \phi l\,\mathbf{W}^{-1}\mathbf{P}^T\mathbf{W} - \frac{m_S m_I}{K}\mathbf{pW}^{-1}\mathbf{Q}^T\mathbf{HQ}\boldsymbol{\beta} \\ & - \frac{m_I(1-m_S)}{K}\mathbf{pW}^{-1}\mathbf{Q}^T\mathbf{H}\boldsymbol{\beta} - \frac{(1-m_I)m_S}{K}\mathbf{pW}^{-1}\mathbf{HQ}\boldsymbol{\beta} \\ & - \frac{(1-m_I)(1 - m_S)}{K}\mathbf{pW}^{-1}\mathbf{H}\boldsymbol{\beta}\Big] = 0\end{aligned}$$

where $\mathbf{p}, \mathbf{H}, \boldsymbol{\beta}, \mathbf{W}, \mathbf{W}^{-1}$ are diagonal, hence commute with each other.

Determinant of $\mathbf{J}^*$

One can prove that $\det(\mathbf{J}^*) = 0 \iff \det(\mathbf{U_3} - \mathbf{G_0}) = 0$

Thus, the the disease-free equilibrium loses stability when $\det(\mathbf{U_3} - \mathbf{G_0}) = 0$, where

$$\mathbf{G_0} = \frac{I}{\mu_B + I}\mathbf{P}^T + \frac{\mu_B}{\mu_B + I}\mathbf{T_0}$$

$\mathbf{T_0} =$
$(1 - m_I)(1 - m_S)\mathbf{R_0} + m_S m_I \mathbf{R_0^{IS}} + m_I(1 - m_S)\mathbf{R_0^{I}} + (1 - m_I)m_S\mathbf{R_0^{S}}.$

Is a transcritical bifurcation happening here?

Stay put!

Theorems

Theorem (Perron-Frobenius)

If A_{nxn} is non-negative and irreducible, then:

1. $\rho(A)$ *is a positive eigenvalue of* A
2. *The eigenvector associated to* $\rho(A)$ *is positive*
3. *The algebraic and geometric multiplicity of* $\rho(A)$ *is 1*

Theorem (Berman, Plemmons)

Let $A = sI - B$, *where* B *is an nxn nonnegative matrix. If there exists a vector* $x > 0$, *such that* $Ax \geq 0$, *then* $\rho(B) \leq s$.

Dominant Eigenvalue

- $\mathbf{G_0} > 0$ and irreducible
- $\rho(\mathbf{G_0})$ is a simple eigenvalue (Perron-Frobenius)
- $\det(\mathbf{U_3} - \mathbf{G_0}) = 0$ implies 1 is an eigenvalue of $\mathbf{G_0}$
- $\rho(\mathbf{G_0}) \leq 1$ (Berman, Plemmons)

Therefore, $\rho(\mathbf{G_0}) = 1$ and 1 is the dominant eigenvalue.

This proves: The disease free equilibrium loses stability when $\rho(\mathbf{G_0})$ crosses one, and an outbreak occurs.

$\rho(\mathbf{G_0})$ is called the Generalized Reproduction Number.

Lyapunov Functions and Global Asymptotic Stability

First write equations as a compartmental system:

$$\mathbf{x}' = \mathbf{F}^*(x,y) - \mathbf{V}^*(x,y), \qquad \mathbf{y}' = g(x,y)$$

where $\mathbf{x} = [I_1 \ I_2 \ I_3 \ B_1 \ B_2 \ B_3]^T$ and $\mathbf{y} = [S_1 \ S_2 \ S_3]^T$.

- $R_0 \leq 1$ does not determine stability for connected communities.
- Cannot use $\rho(\mathbf{G_0})$
- Need to find a new condition.
- Exploited the fact that $\mathbf{J^*} = \mathbf{F} - \mathbf{V}$ to prove:

$$\lambda^*(\mathbf{J^*}) \leq 0 \quad \Leftrightarrow \quad \tilde{\lambda}^*(\mathbf{V}^{-1}\mathbf{J^*}) \leq 0 \quad \Leftrightarrow \quad \lambda_F^*(\mathbf{V}^{-1}\mathbf{F}) \leq 1$$

The three matrices above are Metzler!

Lyapunov Functions and Global Asymptotic Stability

Theorem

Let $0 < \lambda_F^*(\mathbf{V}^{-1}\mathbf{F}) \leq 1$, and let $\mathbf{w}$ be a left eigenvector of $\mathbf{V}^{-1}\mathbf{F}$ corresponding to $\lambda_F^*(\mathbf{V}^{-1}\mathbf{F})$. Then the function $Q(\mathbf{x}) = \mathbf{w}^T\mathbf{V}^{-1}\mathbf{x}$ is a Lyapunov function satisfying $Q' \leq 0$.

$$
\begin{aligned}
Q \quad = \quad & \frac{1}{K\phi(\mu_B+l)\lambda_F^*}\left[I_1\left(\frac{p_1(1-m_l)}{W_1} + \frac{p_2 m_l Q_{12}}{W_2} + \frac{p_3 m_l Q_{13}}{W_3}\right) + \right. \\
& + I_2\left(\frac{p_1 m_l Q_{21}}{W_1} + \frac{p_2(1-m_l}{W_2} + \frac{p_3 m_l Q_{23}}{W_3}\right) + \\
& \left. + I_3\left(\frac{p_1 m_l Q_{31}}{W_1} + \frac{p_2 m_l Q_{32}}{W_2} + \frac{p_3(1-m_l)}{W_3}\right)\right] + \\
& + \frac{1}{\mu_B+l}(B_1 + B_2 + B_3)
\end{aligned}
$$

$Q(\mathbf{x}_0) = Q(\mathbf{0}, \mathbf{0}, \mathbf{H}) = 0$ and $Q(\mathbf{x}) > 0$ when $\mathbf{x} \neq \mathbf{x}_0$.

$Q' = (\lambda_F^* - 1)\mathbf{w}^T\mathbf{x} - \mathbf{w}^T\mathbf{V}^{-1}\mathbf{f}(x, y) \leq 0$

Lyapunov Functions and Global Asymptotic Stability

Theorem (LaSalle's Invariance Principle)

Let $\Omega \subset D \subset R^n$ be a compact invariant set with respect to $x' = f(x)$. Let $Q : D \to R$ be a C^1 function such that $Q'(x(t)) \leq 0$ in Ω. Let $E \subset \Omega$ be the set of all points in Ω where $Q'(x) = 0$. Let $M \subset E$ be the largest invariant set in E. Then

$$\lim_{t \to \infty} \left[\inf_{y \in M} \|x(t) - y\| \right] = 0.$$

That is, every solution starting in Ω approaches M as $t \to \infty$.

Lyapunov Functions and Global Asymptotic Stability

Using LaSalle's invariance principle we can now prove global asymptotic stability.

Theorem

Let Ω be any compact invariant set containing the disease-free equilibrium point of the compartmental model and let $\mathbf{F}$, $\mathbf{V}$, $\mathbf{V}^{-1} \geq 0$, $\mathbf{f}(x, y) \geq 0$, *and* $\mathbf{f}(0, y) = 0$. *Assume* $0 < \lambda_F^*(\mathbf{V}^{-1}\mathbf{F}) < 1$ *and assume the disease-free system* $\mathbf{y}' = g(0, y)$ *has a unique equilibrium* $\mathbf{y} = \mathbf{y}_0 > 0$ *that is globally asymptotically stable in the disease-free system. Then, the disease-free equilibrium of the nine-dimensional system is globally asymptotically stable in Ω.*

Lyapunov Functions and Global Asymptotic Stability

Solving for $\mathbf{S}$ explicitly gives $\mathbf{S} = \begin{bmatrix} H_1 + e^{-\mu t} C_1 \\ H_2 + e^{-\mu t} C_2 \\ H_3 + e^{-\mu t} C_3 \end{bmatrix}$.

$Q' = (\lambda_F^* - 1)\mathbf{w}^T\mathbf{x} - \mathbf{w}^T\mathbf{V}^{-1}\mathbf{f}(x, y)$. Assuming $Q' = 0$ implies $\mathbf{x} = 0$. Then, the set of all points where $Q' = 0$ is $E = \{(I_i, B_i, S_i) : I_i = B_i = 0\}$.

Applying LaSalle's invariance principle shows that the disease-free equilibrium is globally asymptotically stable.

Existence of Endemic Equilibrium

Recall that $\lambda^*(\mathbf{J}^*) \leq 0 \quad \Longleftrightarrow \quad \lambda_F^*(\mathbf{V}^{-1}\mathbf{F}) \leq 1.$

Theorem

Let Ω be any compact invariant set containing the disease-free equilibrium point of compartmental model. Let $\mathbf{F}$, $\mathbf{V}$, $\mathbf{V}^{-1} \geq 0$, $\mathbf{f}(x, y) \geq 0$, and $\mathbf{f}(0, y) = 0$. If $\lambda^*(\mathbf{J}^*) > 0$, then there exists at least one endemic equilibrium.

Hydrological Transportation and Human Mobility

$$P_{ij} = \begin{cases} \dfrac{P_{out}}{d_{out}(i)P_{out} + d_{in}(i)P_{in}} & if \;\; i \to j \\ \dfrac{P_{in}}{d_{out}(i)P_{out} + d_{in}(i)P_{in}} & if \;\; i \leftarrow j \\ 0 & otherwise \end{cases}$$

$$Q_{ij} = \frac{H_j \, e^{(-d_{ij}/D)}}{\sum_{k \neq i}^{n} H_k \, e^{(-d_{ik}/D)}}$$

Geography of Disease Onset

$$\mathbf{J}^* \begin{bmatrix} \mathbf{i} \\ \mathbf{b} \end{bmatrix} = \lambda^* \begin{bmatrix} \mathbf{i} \\ \mathbf{b} \end{bmatrix}$$

$$\mathbf{J}^* = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

$$\begin{aligned} \mathbf{Ai} + \mathbf{Bb} &= \lambda^* \mathbf{i} \\ \mathbf{Ci} + \mathbf{Db} &= \lambda^* \mathbf{b} \end{aligned}$$

$\mathbf{i} = -\mathbf{A}^{-1}\mathbf{Bb}$ and we can re-write the above expression as $\mathbf{CA}^{-1}\mathbf{Bb} + \mathbf{Dd} = 0$

$$(\mathbf{AD} - \mathbf{CB})\mathbf{b} = 0$$

Geography of Disease Onset

After MUCH simplification, we get

$$\mathbf{AD} - \mathbf{CB} = \phi(\mu_B + l)\left[\mathbf{U_3} - \mathbf{W}^{-1}\left(\frac{l}{\mu_B + l}\mathbf{P}^T + \frac{\mu_B}{\mu_B + l}\mathbf{T_0}\right)\mathbf{W}\right]$$

Substituting $\mathbf{G_0}$, we get

$$\mathbf{AD} - \mathbf{CB} = \phi(\mu_B + l)(\mathbf{U_3} - \mathbf{W}^{-1}\mathbf{G_0}\mathbf{W}).$$

We can now write $(\mathbf{U_3} - \mathbf{W^{-1}}\mathbf{G_0}\mathbf{W})\mathbf{b} = 0$, then

$$\mathbf{Wb} = \mathbf{G_0}\mathbf{Wb}$$

Thus, $\mathbf{Wb}$ is the eigenvector of $\mathbf{G_0}$ associated to the eigenvalue $\lambda = 1$.

Geography of Disease Onset

$$\mathbf{i} = \mathbf{A}^{-1}\mathbf{B}\mathbf{b}$$

After substitution and simplification we get,

$$\mathbf{i} = \frac{m_s\mathbf{H}\mathbf{Q}\beta + (1 - m_s)\mathbf{H}\beta}{\phi}(\mathbf{W}^{-1}\mathbf{g_0})$$

Bifurcations

- Recall that $\mathbf{J}^*$ determines stability of disease-free equilibrium, and is a Metzler matrix.
- When $\lambda^* = 0$ stability is lost.
- Proved existence of transcritical bifurcation using Sotomayor's theorem.
- Choose p_1 to be our varying parameter.

Sotomayor's Theorem

Let $\mathbf{f} = [f_1 \quad f_2 \quad \cdots \quad f_9]^T$, and $\mathbf{x} = (S_1 \ S_2 \ S_3 \ I_1 \ I_2 \ I_3 \ B_1 \ B_2 \ B_3)^T$ so $\mathbf{x}_0 = (H_1 \ H_2 \ H_3 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)^T$.

Take partial derivatives of $\mathbf{f}$ with respect to p_1 to arrive at the vector

$$\mathbf{f}_{p_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \left[(1 - m_I)I_1 + m_I(Q_{21}I_2 + Q_{31}I_3)\right] W_1^{-1} \\ 0 \\ 0 \end{bmatrix}.$$

First condition : $\mathbf{w}^T \mathbf{f}_{p_1}(\mathbf{x}_0, p_1^0) = 0$

Sotomayor's Theorem

Second condition:

$$\mathbf{w}^T[D\mathbf{f}_{p_1}(\mathbf{x}_0, p_1^0)\mathbf{v}] =$$
$$\frac{H_1}{\phi}\left[(1-m_S)\beta_1 + m_S Q_{12}\beta_2 + m_S Q_{13}\beta_3\right](1-m_I)W_1^{-1} +$$
$$\frac{H_2}{\phi}\left[m_S Q_{21}\beta_1 + (1-m_S)\beta_2 + m_S Q_{23}\beta_3\right] m_I Q_{21} W_1^{-1} +$$
$$\frac{H_3}{\phi}\left[m_S Q_{31}\beta_1 + m_S Q_{32}\beta_2 + (1-m_S)\beta_3\right] m_I Q_{31} W_1^{-1} \neq 0$$

Sotomayor's Theorem

$D^2\mathbf{f}(\mathbf{x}_0, p_1^0)(\mathbf{v}, \mathbf{v})$ is found.

Third condition: $\mathbf{w}^T[D^2\mathbf{f}(\mathbf{x}_0, p_1^0)(\mathbf{v}, \mathbf{v})] =$

$$w_4(2(1-m_S)\beta_1 v_1 + 2m_S Q_{12}\beta_2 v_1 + 2m_S Q_{13}\beta_3 v_1$$

$$-\ 2H_1(1-m_S)\beta_1 - 2H_1 m_S Q_{12}\beta_2 - 2H_1 m_S Q_{13}\beta_3)$$

$$+\ w_5(2m_S Q_{21}\beta_1 v_2 + 2(1-m_S)\beta_2 v_2 + 2m_S Q_{23}\beta_3 v_2$$

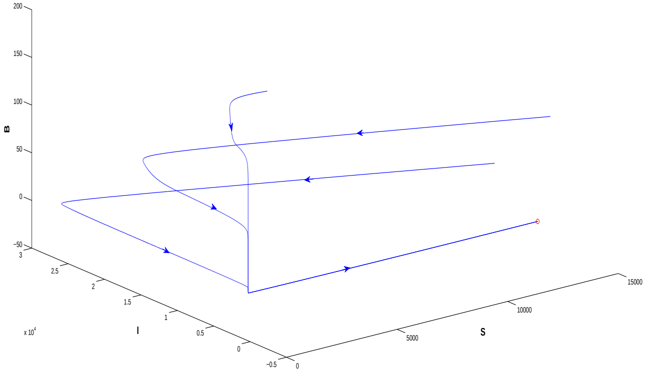
$$-\ 2H_2 m_S Q_{21}\beta_1 - 2H_2(1-m_S)\beta_2 - 2H_2 m_S Q_{23}\beta_3)$$

$$+\ w_6(2m_S Q_{31}\beta_1 v_3 + 2m_S Q_{32}\beta_2 v_3 + 2(1-m_S)\beta_3$$

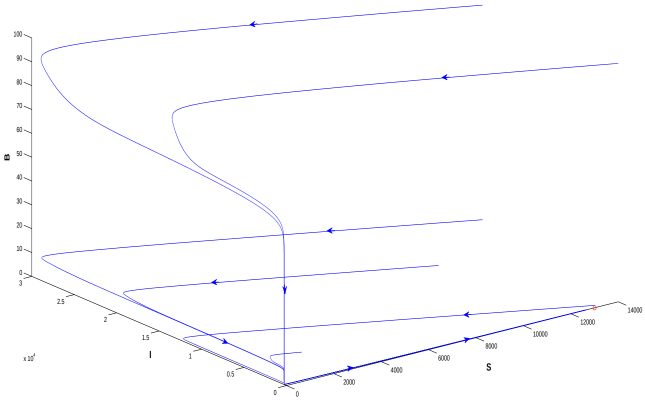
$$-\ 2H_3 m_S Q_{31}\beta_1 - 2H_3 m_S Q_{32}\beta_2 - 2H_3(1-m_S)\beta_3) \neq 0$$

$p < p_0$

DFE =(10000, 13000, 11000, 0, 0, 0, 0, 0, 0)

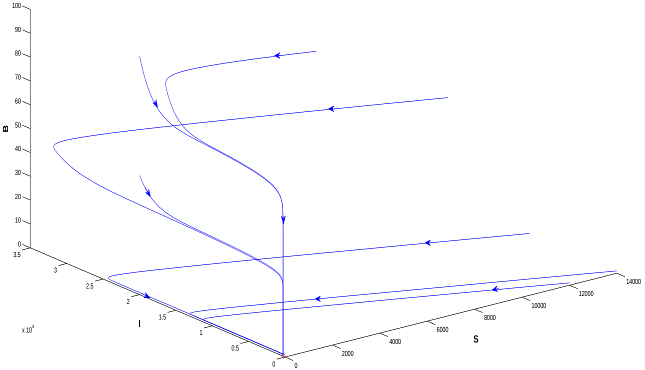


$\mathbf{p} = \mathbf{p}_0$



$\mathbf{p} > \mathbf{p}_0$

$$EE = (11.195, 14.708, 12.952, 215.439, 280.068, 236.969, 0.135, 0.097, 0.157)$$



n Number of Communities

The Jacobian at the disease-free equilibrium $x_0 = (H_1, H_2, \dots, H_n, 0, \dots, 0)$ is given by

$$\mathbf{J}(\mathbf{x}_0) = \begin{bmatrix} \mathbf{J}_{11} & 0 & \mathbf{J}_{13} \\ 0 & \mathbf{J}_{22} & \mathbf{J}_{23} \\ 0 & \mathbf{J}_{32} & \mathbf{J}_{33} \end{bmatrix},$$

where each $\mathbf{J}_{ij}$ block is an $n \times n$ matrix, and

$$\mathbf{J}_{11} = -\mu \mathbf{U_n}$$

$$\mathbf{J}_{13} = -m_S \mathbf{HQ}\boldsymbol{\beta} - (1 - m_S)\mathbf{H}\boldsymbol{\beta}$$

$$\mathbf{J}_{22} = -\phi \mathbf{U_n}$$

$$\mathbf{J}_{23} = m_S \mathbf{HQ}\boldsymbol{\beta} + (1 - m_S)\mathbf{H}\boldsymbol{\beta}$$

$$\mathbf{J}_{32} = \frac{m_I}{K}\mathbf{pW^{-1}Q^T} + \frac{1 - m_I}{K}\mathbf{pW^{-1}}$$

$$\mathbf{J}_{33} = -(\mu_B + l)\mathbf{U_n} + l\mathbf{W^{-1}P^TW}$$

Math Commandments

1. The disease-free equilibrium loses stability when $\lambda^*(\mathbf{J^*})$ crosses zero.

2. The system undergoes a transcritical bifurcation when $\lambda^*(\mathbf{J^*})$ crosses zero.

3. The condition $\det(\mathbf{J^*}) = 0$ is equivalent to $\det(\mathbf{U_n} - \mathbf{G_0}) = 0$.

4. $\lambda = 1$ is the dominant eigenvalue of $\mathbf{G_0}$.

5. If $0 < \lambda_F^*(\mathbf{V}^{-1}\mathbf{F}) \leq 1$, then $Q = \mathbf{w}^T\mathbf{V}^{-1}\mathbf{x}$ is a Lyapunov function satisfying $Q' \leq 0$.

6. If $0 < \lambda_F^*(\mathbf{V}^{-1}\mathbf{F}) < 1$ and the disease-free equilibrium is globally asymptotically stable in the disease-free system, then it is a globally asymptotically stable equilibrium of the general system.

Math Commandments Cont.

7. If $\lambda^*(\mathbf{J^*}) > 0$, then the system has at least one endemic equilibrium point.
8. The dominant eigenvector of $\mathbf{G_0}$, once projected onto the subspace of infectives, provides us with an effective way to forecast the geographical spread of the disease.
9. Under some conditions, no periodic orbits can exist (n=1).
10. The system has no Hopf bifurcations at DFE (n=1).

Our Contribution

- $n = 1$
  - Found correct endemic equilibrium point
  - Proved global stability of DFE
  - Proved existence of transcritical bifurcation
  - (Non)existence of periodic solutions
- $n > 1$ (Networked Connectivity Model)
  - Established a new condition for outbreak of epidemics
  - Introduced an appropriate Lyapunov function
  - Proved global stability of DFE
  - Proved existence of transcritical bifurcation
  - Proved existence of endemic equilibrium
  - Numerical evidence of homoclinic orbit

Back to the Future:

- Proving the existence of the homoclinic orbit.
- Conditions for global stability of endemic equilibrium.
- Hopf bifurcation at endemic equilibrium.
- Get help from XPPAUT to find some other possible bifurcations.
- Include seasonal behavior of some diseases.
- Include water treatment/sanitation and vaccination in the model
- Lunch at Union Club: Chicken Parmesan
- Come back to MSU for MAKO conference!!!!


The End