Genera of Subgroup Intersection Graphs

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Contents

A Recap on Groups

- What is a Group again?
- The **order** of a group G is the number of elements in G, denoted $|G|$.
- The **order of an element** $g \in G$ is the smallest positive integer n such that $g^n = 1$ in G.
- **Subgroups** of G
- **Proper Subgroups** of G: Subgroups that are not the entire group G.

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- A **graph** is a collection of vertices V and a collection of edges E which connect the vertices.
- Typically, vertices are represented as points and edges are represented as lines between those points.
- A **Complete Graph on n vertices** K_n is a graph where every vertex is uniquely connected by an edge.
- K_n has n vertices and $\binom{n}{2}$ edges.

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Genus Formulas

- For an arbitrary graph Γ:

  $\gamma(\Gamma) \geq \lceil \frac{E}{6} - \frac{V}{2} + 1 \rceil$

  $\widetilde{\gamma}(\Gamma) \geq \lceil \frac{E}{3} - V + 2 \rceil$

- For a complete graph K_n:

  $\gamma(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$

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Hasse Diagram of a Group

- The Hasse Diagram of a given group G is the graph whose vertices are the subgroups of G
  and whose edges are determined by "Immediate Inclusion".
- Given $H_1, H_2 \leq G$, we connect H_1 and H_2 with an edge if $H_1 \leq H_2$ and there does not
  exist a subgroup H such that $H_1 < H < H_2$.
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General Strategy

- Since the lower bound for the genus of a complete graph is exact, we look for complete graphs in the subgroup intersection graph of a given group.
- If we can find one that is larger than genus 1, then we are done!
- If the intersection graph is a union of complete subgraphs, we can use the Inclusion-Exclusion principle.
- Inclusion-Exclusion Principle: For two sets A and B, $|A \cup B| = |A| + |B| - |A \cap B|$.
- If we cannot find a subgraph greater than genus 1 we can also explicitly embed the subgroup intersection graph onto a torus or projective plane.
- We have stronger tools which will be seen later.

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An Example

- Recall that a **Cyclic Group** is a group that can be generated by a single element.
- We denote a (finite) Cyclic Group of order n by C_n.
- The **Fundamental Theorem of Cyclic Groups** states that:

1.) Every subgroup of a cyclic group is cyclic, and
2.) There is a one-to-one correspondence between subgroups of C_n and the divisors of n.

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An Example: C_{p^3q}

Normal Subgroup

- A subgroup H of a group G is called a **normal subgroup** of G if $aH = Ha$ for all a in G.

Quotient Groups

- Given a subgroup H of G, we consider sets of the form $\{aH \mid a \in G\}$. These sets partition G into $|G|/|H|$ disjoint classes.
- These sets form a group $G/H = \{aH \mid a \in G\}$ under the operation $(aH)(bH) = (ab)H$, which is well-defined when H is normal in G.
- The **index** of a subgroup, the number of disjoint sets in the partition, is equal to the order of the G divided by the order of H. Intuitively, the index is the "relative size" of H in G.

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Sylow Subgroups

- Let G be a group and let p be a prime. If p^k divides $|G|$ and p^{k+1} does not divide $|G|$, then any subgroup of G of order p^k is called a **Sylow p-subgroup** of G.
- **Sylow's First Theorem** states that there must exist at least one subgroup of order p^k if p^k divides $|G|$.

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Direct and Semi-Direct Products

- Let G and H be groups. We define the direct product of G and H as $H \times G = \{(h, g) \mid h \in H, g \in G\}$, with the operation of $H \times G$ defined coordinate-wise.
- A semi-direct product is a generalization of the direct product. We say a group G is a semi-direct product of a normal subgroup H and subgroup K denoted $H \rtimes K$ if H and K intersect trivially and $G = HK$. If H and K are both normal, then G is the direct product of H and K.

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An Example: *Fred*$_0$

- $(C_p \rtimes C_p) \times C_q = \langle a, b, c \mid a^p = b^p = c^q, cac^{-1} = a^i, cb = bc, ab = ba, ord_p(i) = q \rangle$ and $p > q$.
- Subgroups of order p^2: $\langle a, b \rangle$
- Subgroups of order pq: $\langle a, c \rangle, \langle bc \rangle, \langle b(ac) \rangle, ..., \langle b(a^{p-1}c) \rangle$
- Subgroups of order p: $\langle b \rangle, \langle a \rangle, \langle ab \rangle, ..., \langle a^{p-1}b \rangle$
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Basic Strategy

- We look at groups whose orders have more and more prime factors until, hopefully, they all have genus greater than 1.

The Three Essential Techniques

- If we are trying to show that the genus of the intersection graph of a group G is larger than 1, we can:
  1.) Find a subgroup of G whose intersection graph has genus greater than 1.
  2.) Find a quotient group G/N with genus greater than 1.
  3.) When all else fails, actually find the subgroups of G, and draw out all or part of the Hasse diagram!

Our Favorite Tool

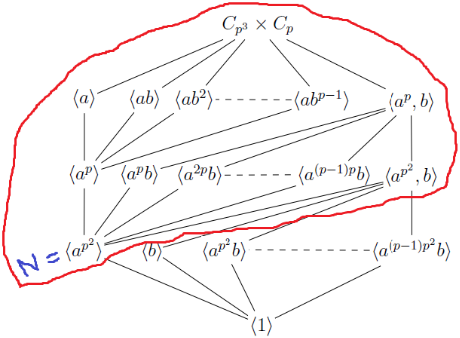
- **The Lattice Isomorphism Theorem**: If N is a normal subgroup of a group G, then there exists a bijection from the set of all subgroups H of G such that H contains N, onto the set of all subgroups of the quotient group G/N. The structure of the subgroups of G/N is exactly the same as the structure of the subgroups of G containing N, with N collapsed to the identity element.

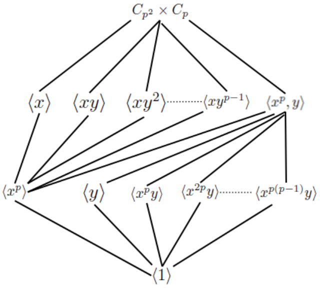
In Short

- So what does this mean? It means that the intersection graph of G/N will look exactly the same as the part of the graph of G that's above the vertex labeled N.

Example

- $C_{p^3} \times C_p / N \cong C_{p^2} \times C_p$ for $N \cong C_p$, i.e. $C_{p^3} \times C_p$ has $C_{p^2} \times C_p$ as a quotient group.





A Nice Consequence of This Theorem

- Notice that if G/N has n proper subgroups, then the Lattice Isomorphism Theorem gives us a K_n subgraph in the intersection graph of G.
- In particular, if G/N has 8 or more proper subgroups, then there will be at least a K_8 in the intersection graph of G, making its genus greater than 1.

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The First Step

- We start by dividing finite groups into two categories: **solvable** and **nonsolvable**.

Solvable Groups

- A group G is said to be **solvable** if we can write $\langle 1 \rangle = H_0 \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_{n-1} \trianglelefteq H_n = G$ where the order $|H_{i+1}/H_i|$ is prime for all i.
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Fun Facts About Solvable Groups!

- The chain of normal subgroups tells us that the order of G/H_{n-1} is prime. So the order of H_{n-1} has one prime factor less than the order of G.
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Strategy For Solvable Groups

- For example if $|G| = p^4q^2r$, then G has a subgroup of order p^3q^2r, p^4qr, or p^4q^2.
- If we have shown that all such groups have genus greater than 1, then G is automatically eliminated.

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Strategy For Solvable Groups

- We can use this fact to find large quotient groups of a given group G.
- For example, if $|G| = p^2q^2r$, then $|N| = p, p^2, q, q^2, r$ or r^2.
- So $|G/N| = |G|/|N| = p^2q^2, pq^2r, q^2r, p^2qr$, or p^2r.
- Very few of these groups have fewer than 7 proper subgroups, so we have narrowed down the possibilities substantially.

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Table of Subgroups

Number of Proper Subgroups	Groups
1	C_p
2	C_{p^2}
3	C_{pq}, C_{p^3}
4	$C_2 \times C_2, C_{p^4}$
5	$S_3, Q_8, C_3 \times C_3, C_{p^2q}, C_{p^5}$
6	C_{p^6}
7	$C_4 \times C_2, D_{10}, C_3 \rtimes C_4, C_5 \times C_5, C_{pqr}, C_{p^3q}, C_{p^7}$

Fun Facts About Solvable Groups!

- And the very best fun fact: Any solvable group whose order has more than 3 distinct prime factors is automatically eliminated; its intersection graph will always have genus greater than 1.

Proof

- Let G be a solvable group of order $p^\alpha q^\beta r^\delta s^\gamma \cdots$. The Sylow Theorems guarantee that G has subgroups P, Q, R, S, of orders $p^\alpha, q^\beta, r^\gamma$, and s^δ, respectively.
- Since G is solvable, these form a Sylow Basis; the product of any set of these subgroups is itself a subgroup. For example, PQ and PQS are subgroups of G.

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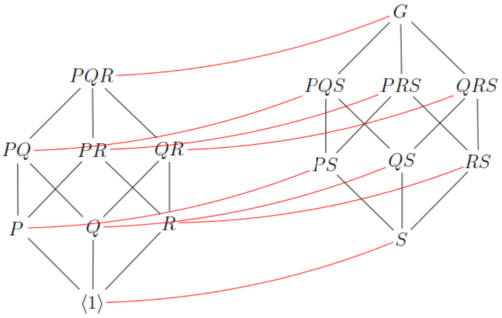
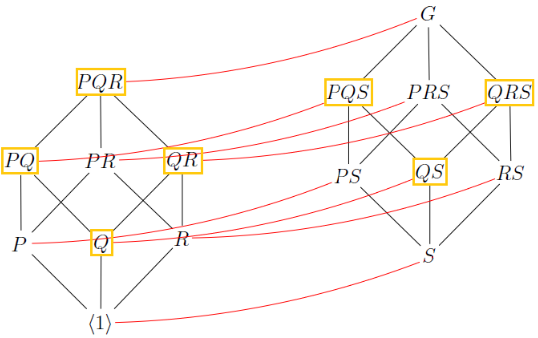
Proof

- This gives us the following portion of the Hasse diagram of G:



Proof

- We see that Q is contained in six other proper subgroups:



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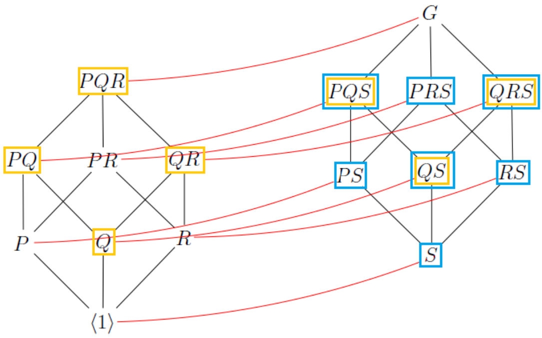
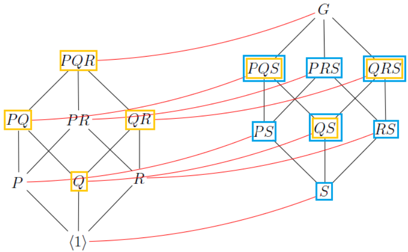


Proof

- This will produce two copies of K_7 meeting at three vertices in the intersection graph of G. We write $K_7 \vee_{K_3} K_7 \subseteq \Gamma(G)$.
- This subgraph has $\binom{7}{2} + \binom{7}{2} - 3 = 39$ edges and 11 vertices by Inclusion-exclusion.
- It has genus at least $\gamma(K_7 \vee_{K_3} K_7) \geq \lceil \frac{39}{6} - \frac{11}{2} + 1 \rceil = \lceil \frac{12}{6} \rceil = 2$. So G is too big!



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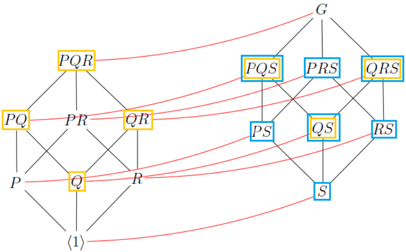

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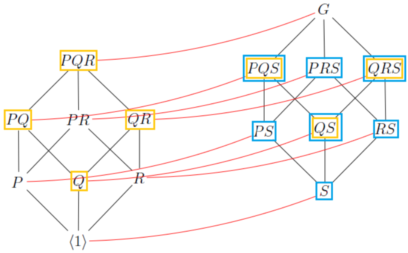

Strategy for Solvable Groups

- Abelian Groups
- $p-$groups
- Groups of order p^2q
- Groups of order $p^\alpha q$
- Groups of order p^2q^2
- Groups of order $p^\alpha q^\beta$
- Groups of order pqr
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What We Are Working On Now

- We are currently working on solvable groups of order p^2q^2 and order p^2qr; so far they all have genus greater than 1, so it looks like we have almost reached the end!

Nonsolvable Groups

- Every nonsolvable group contains a minimal simple group as a subquotient.
- In other words, the Hasse diagram of a non-solvable group contains that of a minimal simple group as a sub-lattice.
- There are essentially five possible minimal simple groups:
- $L_2(2^p)$, $L_2(3^p)$, $L_3(3)$, $L_2(p)$, and $Sz(2^q)$.
- Each of these has a solvable subgroup with genus greater than 1.

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The Future

We hope to finish up the remaining solvable groups; in addition, there are a number of other properties of subgroup intersection graphs that could be explored, including:

- Nonorientable Genus
- Hamiltonian Cycles
- Chromatic Number
- Higher genera
- *And much, much more!*

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