NON-CYCLIC GRAPHS OF FINITE GROUPS

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Presentation Outline

- Introduction
- Eulerian Circuits and Paths
- Hamiltonian Circuits
- Automorphisms
- Small Orientable and Nonorientable Genera
- Future Work

INTRODUCTION

Introduction: Groups

A **group** is a set G with an operation $*$ such that:

- The group is *closed* under $*$.
- $*$ is *associative*.
- G has an *identity* element under $*$.
- Every element of G has an *inverse* under $*$.

Examples: $\mathbb{Z}$ under addition, $\mathbb{R}$ under multiplication, C_5 under modular addition.

Introduction: Non-cyclic graphs

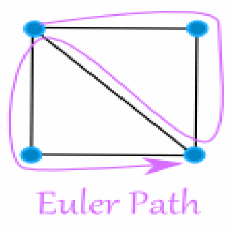
If G is a group, then the **non-cyclic graph** of G is the graph $\Gamma(G)$ defined as follows:

- Vertices are elements of G.
- Connect two vertices if the corresponding elements do *not* generate a cyclic group together. (In other words, x and y are connected iff $\langle x, y\rangle$ is non-cyclic.)
- Remove all vertices that are not connected to anything.

EULERIAN CIRCUITS AND PATHS

Eulerian paths

An **Eulerian path** is a path that uses every edge of a graph exactly once.

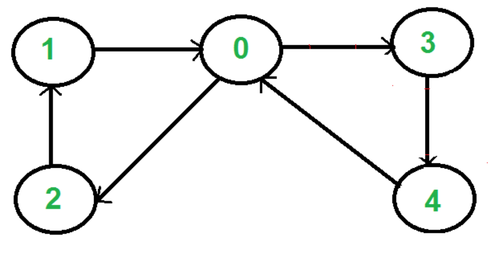


Source: https://www.mathsisfun.com

A graph has an Eulerian path iff precisely two vertices are of odd degree, and all others are of even degree. Such a graph is called **path-Eulerian**.

Eulerian circuits

An **Eulerian circuit** is an Eulerian path that starts and ends at the same vertex.



Source: https://www.math.ku.edu/~jmartin/courses/math105-F11/Lectures/chapter5-part2.pdf

A graph has an Eulerian circuit iff every vertex has even degree. Such a graph is called **Eulerian**.

Eulerian groups

What is the **degree** of an element in $\Gamma(G)$?

The **cyclicizer** of an element x, written $\mathrm{Cyc}_G(x)$, is the set of all $y \in G$ such that $\langle x, y \rangle$ is cyclic.

G is **Eulerian** if $|G|$ and $|\mathrm{Cyc}_G(x)|$ are either both even or both odd for all elements x.

Families of Eulerian groups

The following groups are Eulerian:

- All groups of odd order.
- All abelian groups.
- All 2-groups.

The following are non-Eulerian:

- S_n for $n \geq 3$.
- All nonabelian simple groups.

More results on Eulerian groups

- If H has odd order, then $G \times H$ is Eulerian iff G is Eulerian.
- If H has even order, then $G \times H$ is Eulerian whenever at least one of G or H is Eulerian.
- S_3 is the only path-Eulerian group.

Centralizers

The **centralizer** of x, written $C_G(x)$, is the set of all elements that commute with x.

If G has even order, then $|\mathrm{Cyc}_G(x)|$ and $|C_G(x)|$ always have the same even-odd parity.

Non-commuting graph of a group

If G is a group, then the **non-commuting graph** of G is the graph defined as follows:

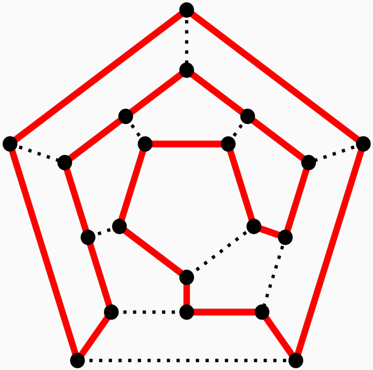
- Vertices are elements of G.
- Connect two vertices if the corresponding elements do not commute.
- Remove all vertices that are not connected to anything.

Eulerianness of non-cyclic and non-commuting graphs is equivalent.

HAMILTONIAN CIRCUITS

Hamiltonian circuits

A **Hamiltonian circuit** is a path that uses every vertex of a graph exactly once and starts and ends at the same vertex.



Source: Wikipedia

The **independence number** $\alpha(\Gamma)$ of a graph Γ is the size of the largest set of mutually non-connected vertices in Γ.

A **quotient group** G/N of a group G is a group obtained by aggregating distinct elements in G into equivalence classes that preserve aspects of the group structure of G.

Example: C_n, the integers modulo n.

Hamiltonian circuits

Theorem (Dirac)

If $\delta(\Gamma) \geq n/2$*, where* $\delta(\Gamma)$ *is the minimum vertex degree in* Γ *and* $n = |\Gamma|$*, then* Γ *is Hamiltonian.*

Theorem

In a non-cyclic graph $\Gamma(G)$*,* $\alpha(\Gamma(G)) < m$*, where* m *is the size of the largest cyclic subgroup of* G*.*

Hamiltonicity of Complete Multipartite $\Gamma(G)$

Theorem

Any complete multipartite non-cyclic graph Γ is Hamiltonian.

Proof.

Since G is not a cyclic group, the largest a cyclic subgroup of G can be is $|G|/2$. This implies that $\alpha(\Gamma) < |\Gamma|/2$. Since Γ is complete multipartite, every vertex in a maximal independent set is connected to every vertex outside of that set, thus $d(x) > |\Gamma|/2$ for any $x \in V(\Gamma)$. □

Hamiltonian circuits

Conjecture

The non-cyclic graph of every non-cyclic group contains a Hamiltonian circuit.

Other Hamiltonicity Theorems

Theorem (Abdollahi & Hassanabadi)

If $\Gamma(G/Cyc(G))$ *is Hamiltonian, then* $\Gamma(G)$ *is Hamiltonian.*

Corollary

If every group with trivial cyclicizer is Hamiltonian, then every non-cyclic group is Hamiltonian.

Theorem (Abdollahi & Hassanabadi)

If $Z(G) = Cyc(G)$*, then* $\Gamma(G)$ *is Hamiltonian.*

AUTOMORPHISMS

Automorphisms of Groups

An **automorphism of a group** G is a bijective function $\phi : G \to G$ that preserves the group structure of G, so that

$$\phi(ab) = \phi(a)\phi(b)$$

for any $a, b \in G$.

A **characteristic subgroup** H of G is a subgroup that is invariant under every automorphism of G. I.e., for ϕ an automorphism of G and any $h \in H$, $\phi(h) = h'$, where $h' \in H$.

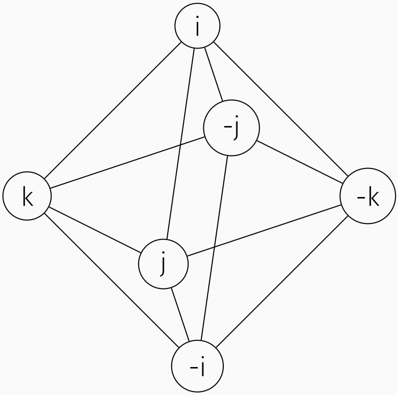
Automorphisms of Graphs

An **automorphism of a graph** Γ is a bijective function $\phi : V(\Gamma) \to V(\Gamma)$ such that if $ab \in E(\Gamma)$, then $\phi(a)\phi(b) \in E(\Gamma)$.

Any automorphism of a group G must induce an automorphism on the non-cyclic graph $\Gamma(G)$.

Automorphisms of Q_8

The quaternion group Q_8 is the group that satisfies the presentation $Q_8 = \langle -1, i, j, k \mid (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle$.



Correspondence between Aut(G) and Aut(Γ)

Theorem

There is no group G such that its non-cyclic graph $\Gamma(G)$ has no non-trivial graph automorphisms.

Theorem

$Cyc(G)$ is a characteristic subgroup of G.

ORIENTABLE AND NONORIENTABLE GENERA

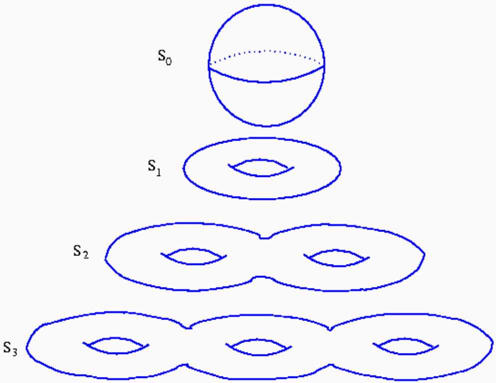
Graph embeddings

The **genus of a graph** is the smallest integer n such that the graph can be embedded (no overlapping points or edges) in a surface of genus n.

The **genus of an orientable surface** refers to its number of “holes”. It is denoted γ.

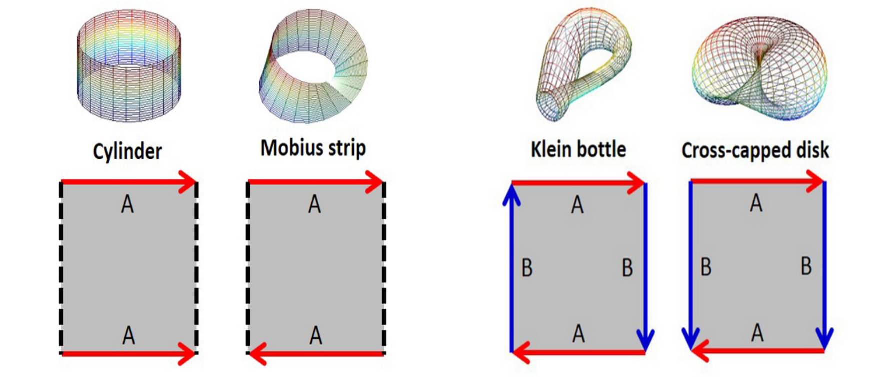
The **non-orientable genus** of a surface, written $\tilde{\gamma}$, refers to its number of cross-caps.

Orientable Genus



Source: http://www.personal.kent.edu/~rmuhamma/GraphTheory/MyGraphTheory/embedding.htm

Non-orientable surfaces



Determining the genus of a graph

If v is the number of vertices and e is the number of edges in a graph, then

$$\gamma \geq \frac{e}{6} - \frac{v}{2} + 1 \quad \text{and}$$
$$\tilde{\gamma} \geq \frac{e}{3} - v + 1.$$

If the RHS is an integer n, then a graph has a genus-n embedding if and only if it is a **triangular embedding**.

Genera of non-cyclic graphs

First, find a lower bound for the number of edges e.

- The probability that two randomly selected elements of a non-cyclic group generate a cyclic subgroup together is

$$1 - \frac{2e}{n^2},$$

  where n is the order of the group.
- This probability is bounded above by $5/8$.
- Therefore, e is bounded below by $3n^2/16$.

Genera of non-cyclic graphs

If G is a non-cyclic group of order n, then

$$\gamma(\Gamma(G)) \geq \left\lceil \frac{n^2 - 16n + 48}{32} \right\rceil \text{ and}$$
$$\tilde{\gamma}(\Gamma(G)) \geq \left\lceil \frac{n^2 - 16n + 48}{16} \right\rceil.$$

If $\gamma \leq 3$ or $\tilde{\gamma} \leq 6$, then $n \leq 19$.

Determining the genus of a graph

- Is there a subgraph we know the genus of? Is the graph itself a subgraph of a well-known graph?
- Does the formula only hold true for triangular embeddings? Can such an embedding exist?
- Can we draw an embedding in a surface of this genus?

Determining the genus of a graph

How do we get from a polygonal configuration to an embedding?

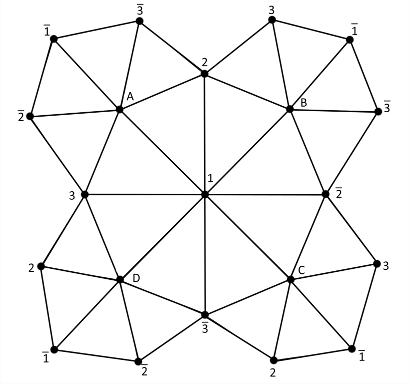


Figure: Dic_{12} nonorientable genus 4 embedding

Figure: Eliminating point $\bar{2}$ from the perimeter

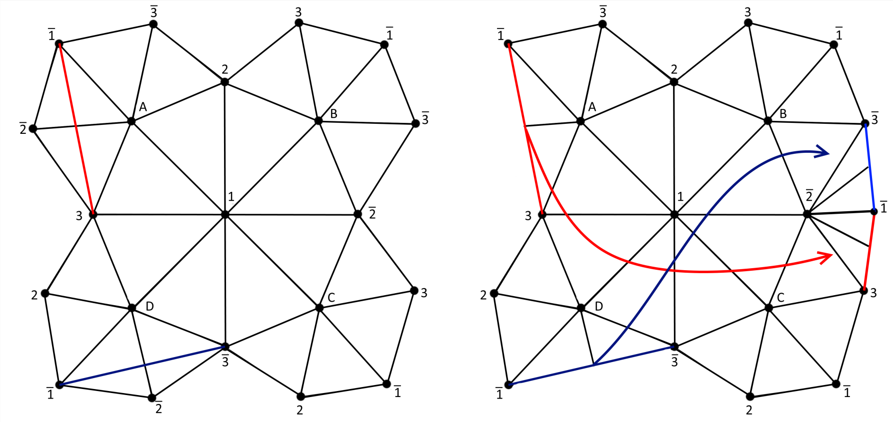
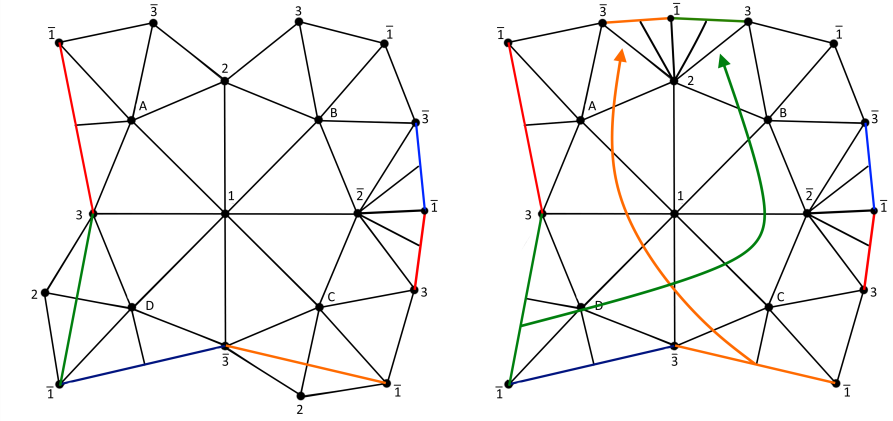


Figure: Eliminating point 2 from the perimeter



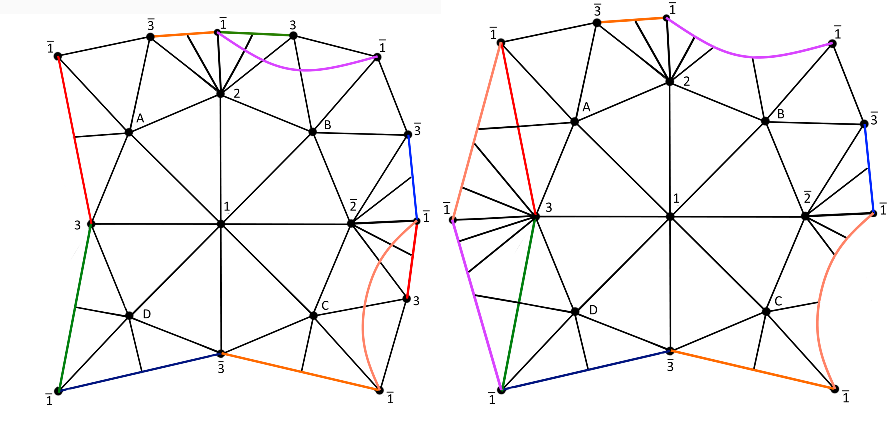


Figure: Eliminating point 3 from the perimeter

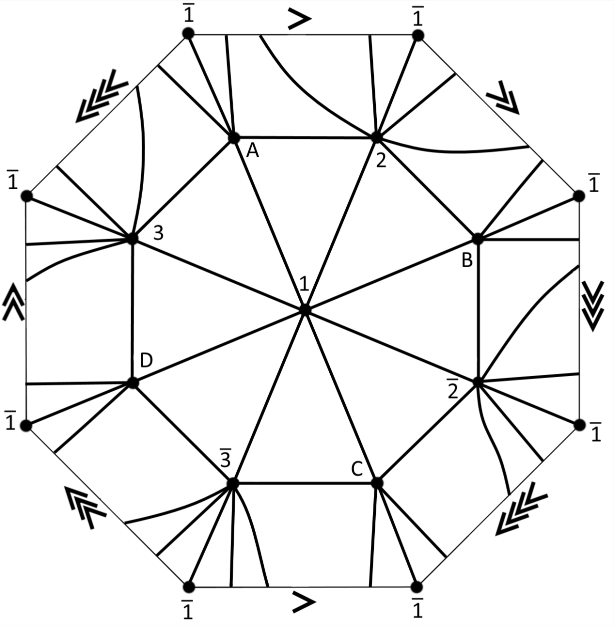


Figure: Dic_{12}, nonorientable genus 4

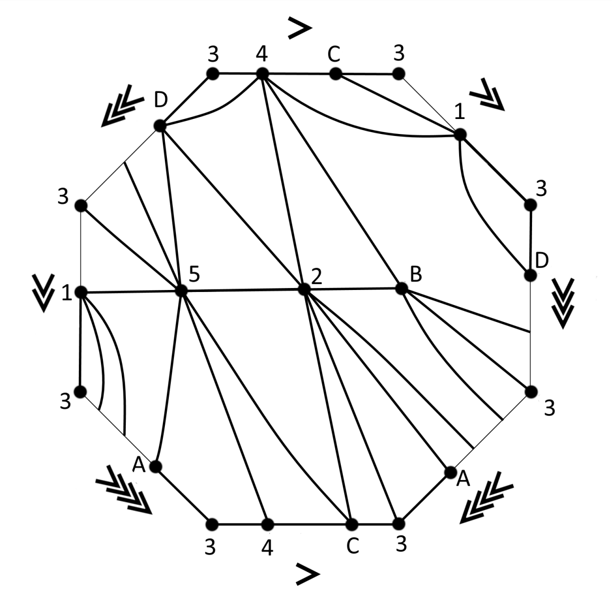


Figure: D_{10}, orientable genus 2

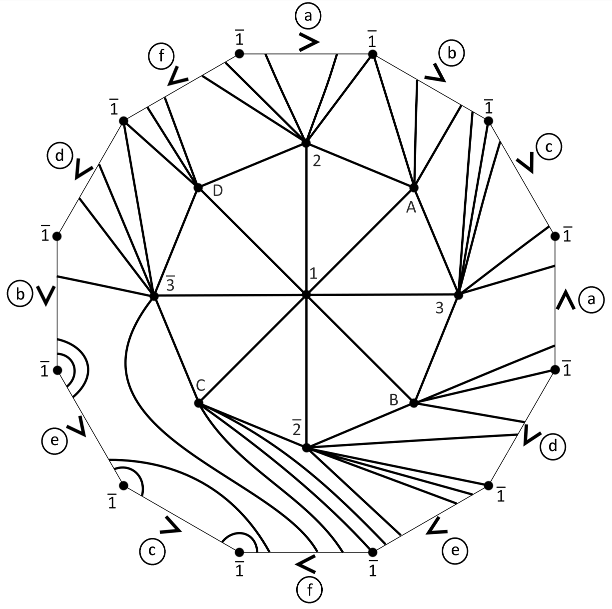


Figure: Dic_{12}, orientable genus 3

Genera of non-cyclic graphs

Group	γ	$\tilde{\gamma}$
$C_2 \times C_2$	0	
S_3	0	
$C_2 \times C_4$	1	1
D_8	1	1
Q_8	0	
C_2^3	1	3
$C_3 \times C_3$	1	3
D_{10}	2	4
Dic_{12}	3	4
D_{12}	≥ 4	6
$C_2 \times C_2 \times C_3$	1	3

FUTURE WORK

Future Work

- Orientable genus 4
- Genera of families of groups
- Further Hamiltonian classification
- Eulerianness of semi-direct products