Dynamics of the Pentagram Map

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Midpoint Iteration

- Problem of midpoint iterations

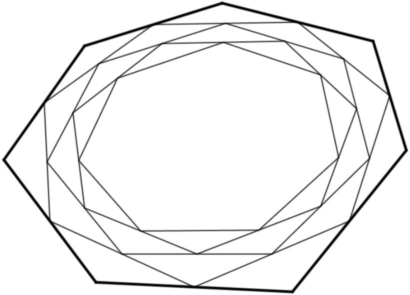

FIGURE – Basic midpoint iteration on a heptagon

Midpoint Iteration

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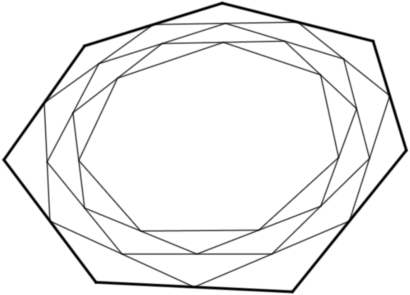
- Finite Fourier Transforms



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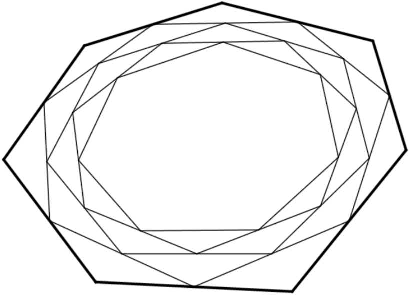

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Midpoint Iteration

- Problem of midpoint iterations
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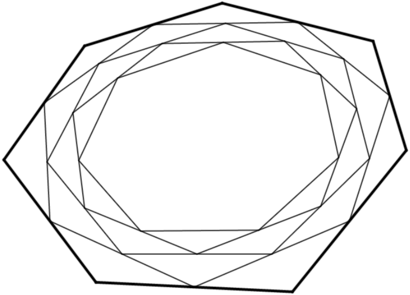

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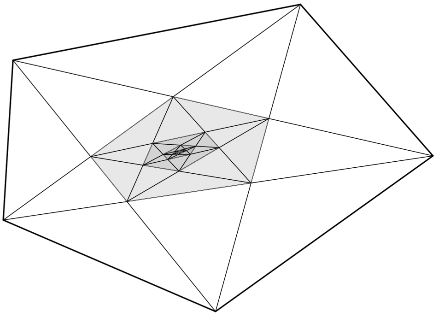
Variation

- First proposed in 1800's



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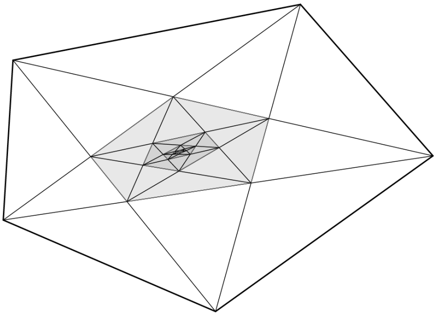

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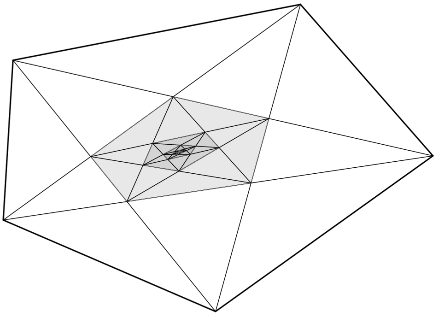

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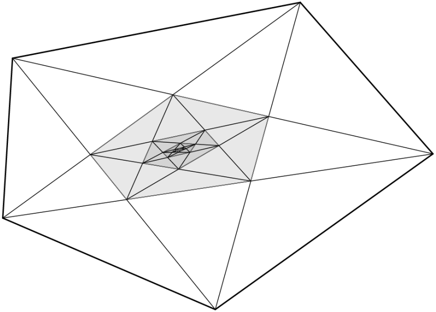

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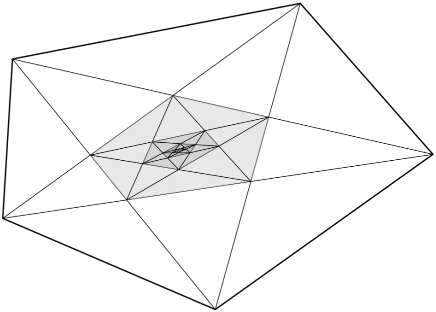
Our Work

- Looked into the generalizations to any n-gon



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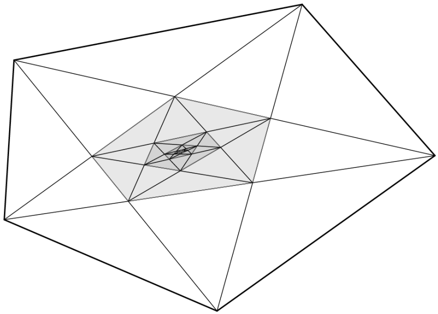

FIGURE – Pentagram mapping on a pentagon

Our Work

- Looked into the generalizations to any *n*-gon
- Shown that the max rate of area decrease for any pentagon is 14/15
- Proved convergence for any regular *n*-gon

Midterm Goals

- Generalize the pentagon's area method to *n*-gons

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- Use the matrices to prove convergence of vertices
- Explore connections to projective geometry

Contents

Schwartz's Proof

Richard Schwartz [4] proved that the pentagram map converges on any convex polygon.

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The cross ratio of collinear points $A, B, C, D \in \mathbb{R}^2$ is defined as

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If the four points are ordered A, B, C, D, then $\chi(A, B, C, D) \geq 1$.

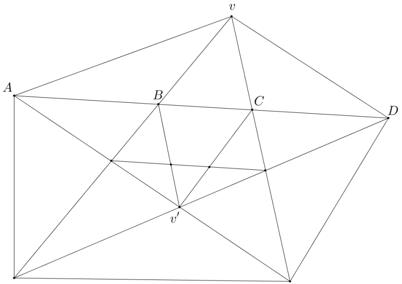
Schwartz's Proof

Definition

Let v be a vertex of a polygon Π. The vertex invariant of v, written $\chi(v)$ is defined by

$$\chi(v) = \chi(A, B, C, D)$$

where A, B, C, and D are the points defined in the diagram.



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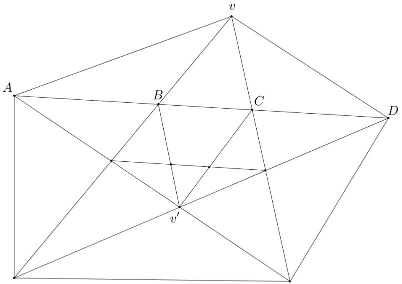

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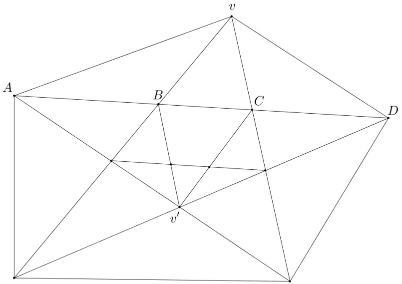


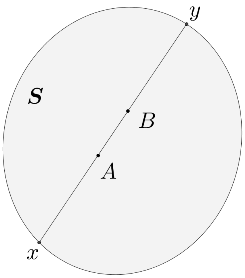
- The pentagram map preserves the vertex invariants of a pentagon.
- The pentagram map preserves the product of the vertex invariants for any polygon.

Schwartz's Proof

Definition

Let $A, B \in S$ where S is a convex subset of $\mathbb{R}^2$. Let x and y be the intersection points of the line through A and B with the boundary of S, where the points are ordered x, A, B, y. Then the Hilbert Distance between A and B in S is defined as

$$\delta_S(A, B) = \log(\chi(x, A, B, y))$$



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3. The Hilbert Distance between the endpoints of $L \bigcap \Pi^k$ inside Π^{k-1} becomes infinite as $k \to \infty$.
4. By the triangle inequality, the Hilbert Perimeter of Π^k in Π^{k-1} becomes infinite as $k \to \infty$.

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4. By the triangle inequality, the Hilbert Perimeter of Π^k in Π^{k-1} becomes infinite as $k \to \infty$.
5. But we can show that the Hilbert Perimeter of Π^k in Π^{k-1} is the log of the product of the vertex invariants of Π^k, so it is invariant with respect to the pentagram map.

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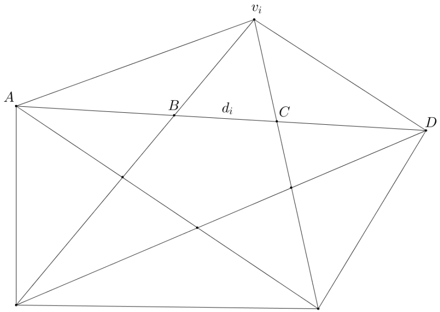
5. But we can show that the Hilbert Perimeter of Π^k in Π^{k-1} is the log of the product of the vertex invariants of Π^k, so it is invariant with respect to the pentagram map.
6. Contradiction !

Convergence for a Restricted Class of Pentagons



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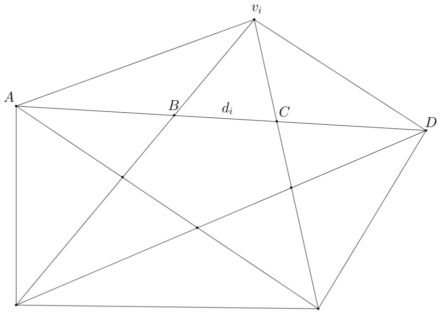

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$\implies |AC| < k_i|AB|$

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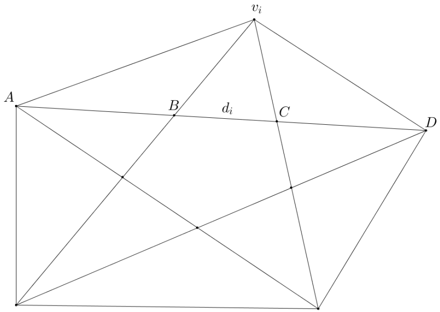

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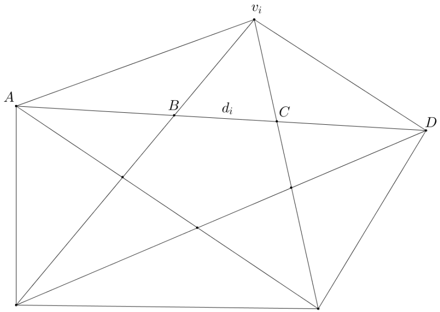

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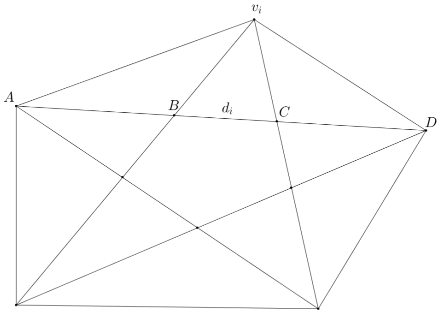

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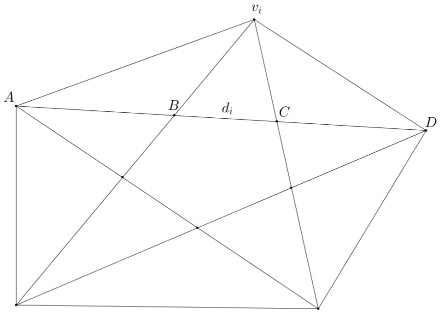

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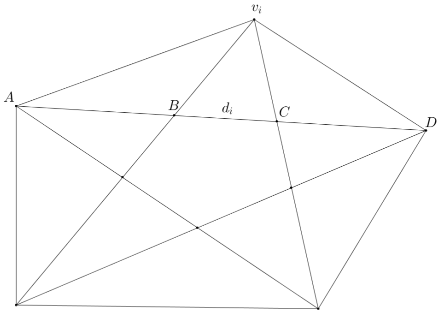

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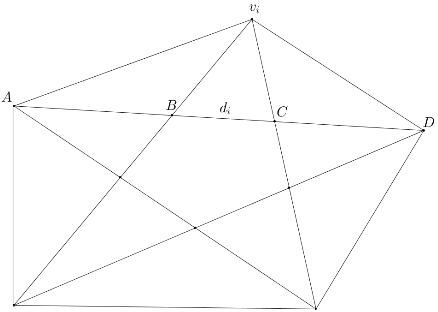

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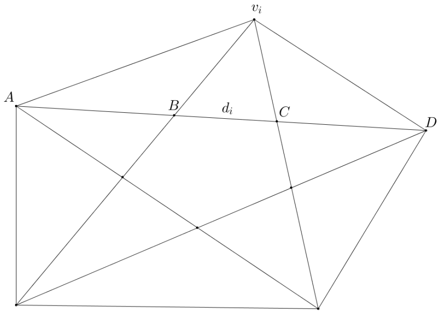

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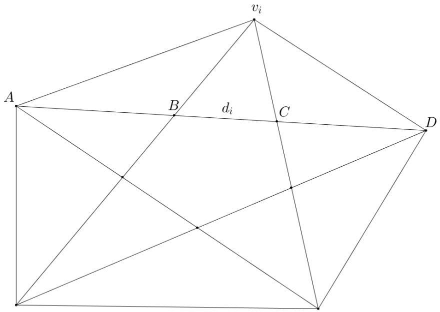

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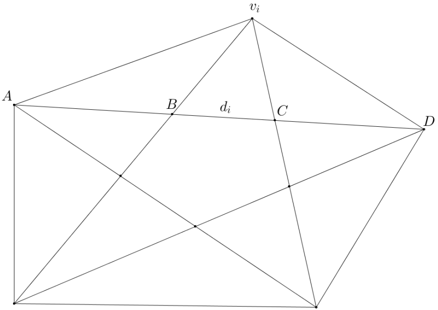

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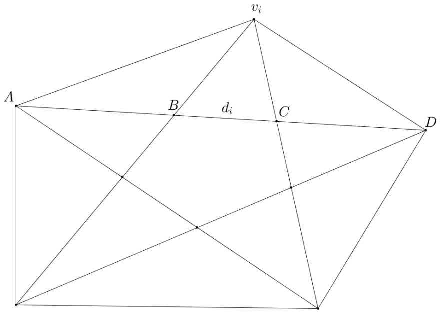

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$P(\Pi^1) < \sum_{i=0}^{4} \left(\frac{k_i - 1}{k_i + 1}\right) d_i$

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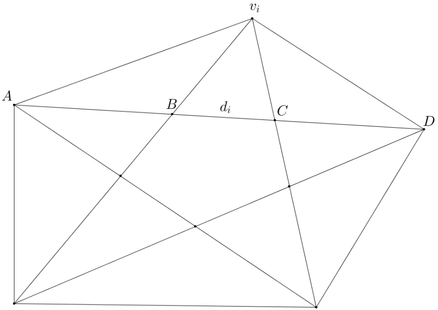

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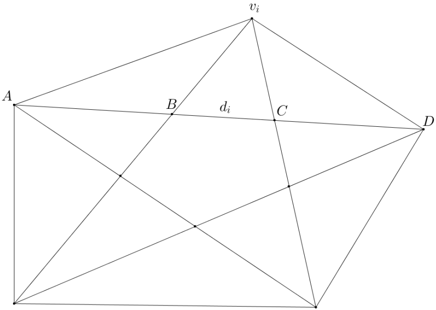

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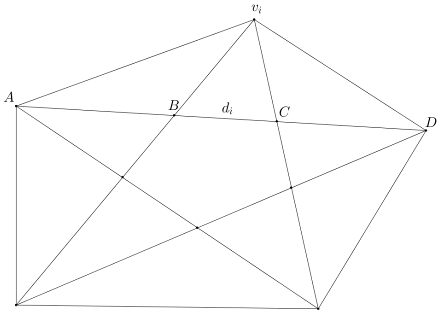

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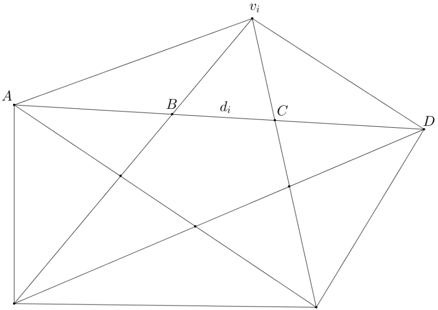

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- The k_i values are invariant under the pentagram map.
- So $\frac{P(\Pi^k)}{P(\Pi^0)} < \left(2\left(\frac{k_{max} - 1}{k_{max} + 1}\right)\right)^k$

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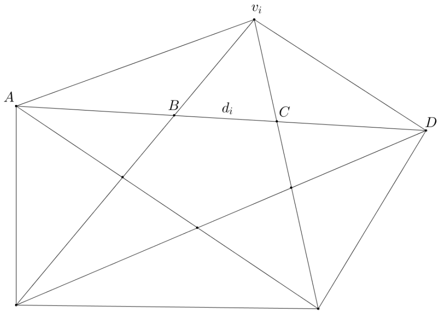

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- So $\frac{P(\Pi^k)}{P(\Pi^0)} < \left(2\left(\frac{k_{max} - 1}{k_{max} + 1}\right)\right)^k$
- If $k_{max} < 3$, then the pentagram iteration converges to a point and we have a bound for the rate.

A Conjecture

Explorations in geogebra indicate that $\frac{P(\Pi^1)}{P(\Pi^0)} < \frac{k_{max}-1}{k_{max}+1}$ holds in general for any polygon, regardless of the number of sides, but we have not been able to prove this.

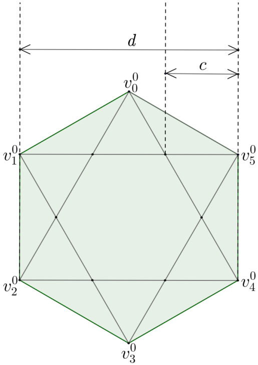
Representation by matrices

The Pentagram map can be represented by an $n \times n$ circulant-patterned matrix.

$$M = \begin{bmatrix} \alpha_0 & 0 & 1-\alpha_0 & 0 & \dots & 0 \\ 0 & \alpha_1 & 0 & 1-\alpha_1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 1-\alpha_{n-1} & 0 & 0 & \dots & \alpha_{n-1} \end{bmatrix}$$

where α_i is a proportion along the i^{th} diagonal, or $\alpha = \frac{c}{d}$

> Note : In a regular n-gon
> $\alpha_0 = \alpha_1 = \dots = \alpha_{n-1} = \left(\frac{\sin(\pi/n)}{\sin(2\pi/n)}\right)^2$



Matrices continued

Multiplying M by a vector of vertices will result in a column vector of the next polygon's vertices

$$\text{vertices of } \Pi^{k+1} = M_k(\text{vertices of } \Pi^k)$$

$$\begin{bmatrix} v_0^{k+1} \\ v_1^{k+1} \\ \vdots \\ v_{n-1}^{k+1} \end{bmatrix} = \begin{bmatrix} \alpha_0 & 0 & 1-\alpha_0 & 0 & \dots & 0 \\ 0 & \alpha_1 & 0 & 1-\alpha_1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 1-\alpha_{n-1} & 0 & 0 & \dots & \alpha_{n-1} \end{bmatrix} \begin{bmatrix} v_0^{k} \\ v_1^{k} \\ \vdots \\ v_{n-1}^{k} \end{bmatrix}$$

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We can then express the vertices of Π^k as

$$\Pi^k = M_k M_{k-1} \dots M_0 \Pi^0$$

Our project's main goal is to show that the vertices of Π^k converge as $k \to \infty$

Past Uses

- Eric Hintikka [1] used coefficients of ergodicity to prove that any polygon derived from a series of stochastic circulant-patterned matrices will converge.
- **Stochastic :** All entries in each row will add to one and be non-negative.
- **Circulant- patterned :** Each matrix has the same zero pattern, which repeats through each row while shifting one column each time.

$$M = \begin{bmatrix} \alpha_0 & 0 & 1-\alpha_0 & 0 & \dots & 0 \\ 0 & \alpha_1 & 0 & 1-\alpha_1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & 1-\alpha_{n-1} & 0 & 0 & \dots & \alpha_{n-1} \end{bmatrix}$$

Coefficients of Ergodicity

Generally, ergodicity coefficients estimate the rate of convergence for stochastic matrices [2].
We'll use some key properties of one coefficient, τ_1 :

1. $0 \leq \tau_1(M) \leq 1$, and $0 = \tau_1(M) \Leftrightarrow M$ is a rank one matrix
2. $\tau_1(M) = 1 - \sum_{k=1}^{n} min\{m_{ik}, m_{jk}\}$
3. $\tau_1(M_1 M_2) \leq \tau_1(M_1)\tau_1(M_2)$

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Scheme :

- For a sequence of k stochastic matrices, divide them into groups of n. Call one such group M_g.

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- Each group will multiply to create a positive, stochastic matrix, with $\tau_1(M) = 1 - \sum_{k=1}^{n} min\{m_{ik}, m_{jk}\}$. Then we know that $\tau_1 < 1$ for each group
  - specifically, we have $\tau_1(M_g) \leq 1 - n\epsilon^{(n-1)}$ where ϵ is the smallest entry in any M matrix that is greater than zero.

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$$\lim_{k\to\infty} \tau_1(M^k) \leq \lim_{k\to\infty} (1 - n\epsilon^{(n-1)})^k$$

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- Which will equal zero when we have a bound on ϵ, the smallest possible α value.
- Which implies M^k is a rank one matrix, say L.
- Thus, the polygon converges, as

$$\lim_{k\to\infty} \Pi^k = L\Pi^0$$

- Which is simply a point.

Limitations

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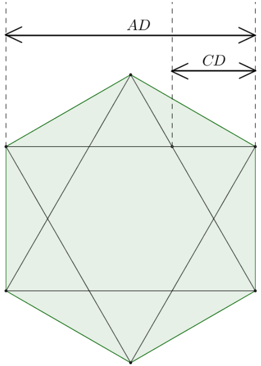
- Method only works for polygons with odd number of sides :

$$M = \begin{bmatrix} \alpha_0 & 0 & 1-\alpha_0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 1-\alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 & 1-\alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 & 0 & 1-\alpha_3 \\ 1-\alpha_4 & 0 & 0 & 0 & \alpha_4 & 0 \\ 0 & 1-\alpha_5 & 0 & 0 & 0 & \alpha_5 \end{bmatrix}$$

$$M^k = \begin{bmatrix} \gamma_0 & 0 & \gamma_1 & 0 & \gamma_2 & 2 \\ 0 & \beta_0 & 0 & \beta_1 & 0 & \beta_2 \\ \phi_2 & 0 & \phi_0 & 0 & \phi_1 & 0 \\ 0 & \psi_2 & 0 & \psi_0 & 0 & \psi_1 \\ \rho_1 & 0 & \rho_2 & 0 & \rho_0 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_0 \end{bmatrix}$$

Possibilities

- If we consider α in terms of the cross-ratio, we would have
$$\alpha = \frac{CD}{AD}$$
- If a bound exists on this α in terms of k, even a restricted case of k, then we are able to use the coefficients of ergodicity



Set-up

Represent the pentagon as a loop :

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- The x-coordinates in the loop representing the pentagon

$$x(t) = \begin{cases} (1-5t)a_0 + 5ta_1 & 0 \leq t \leq 1/5 \\ (2-5t)a_1 + (5t-1)a_2 & 1/5 \leq t \leq 2/5 \\ (3-5t)a_2 + (5t-2)a_3 & 2/5 \leq t \leq 3/5 \\ (4-5t)a_3 + (5t-3)a_4 & 3/5 \leq t \leq 4/5 \\ (5-5t)a_4 + (5t-4)a_0 & 4/5 \leq t \leq 1 \end{cases}$$

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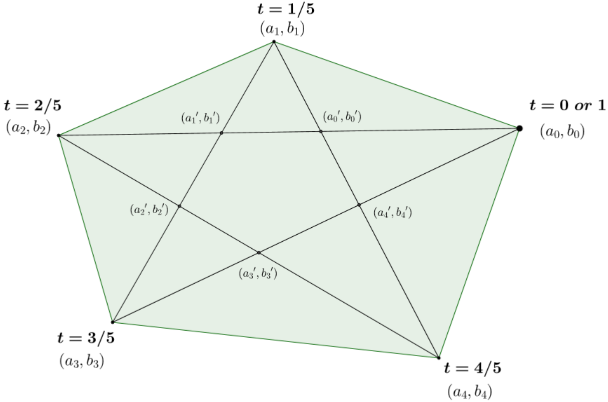
- The y-coordinate parametrization has the same form, except all *a*'s are replaced with *b*.

Loop



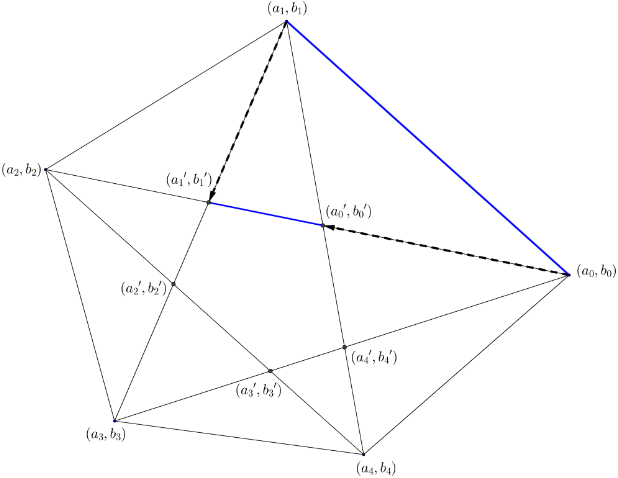
t	$(x(t), y(t))$
0 , 1	(a_0, b_0)
1/5	(a_1, b_1)
2/5	(a_2, b_2)
3/5	(a_3, b_3)
4/5	(a_4, b_4)

FIGURE – The parametrization of the pentagon

Moving Segments

Paramterization of the first pentagon allows for a simple linear translation of segments when defining the second pentagon.

Requires all segments to be defined by a unique linear transformation.



Matrix Multiplication

- Because the transformation is a linear one, we can represent the points through matrix multiplication.

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \qquad \frac{i}{5} \leq t \leq \frac{(i+1)}{5}$$

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- One downside is that a matrix needs to be found for each segment's transformation.

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> However, this is a lot of calculation to do by hand.
>
> BRING IN THE PYTHON !

The Game

```
def pent(file_name , n):    # 'n'= number of iterations to be done
    count = 0
    for itevar in range(0, n):
        p = perimeter(file_name)
        d = diagonal(file_name)
        i = intersection(file_name)
        c1 = count + 1
        write_file(file_name , count)
        new_file_name = str(file_name.replace('_' + str(count) + '.txt', '_' + str(c1) + '.txt'))
        p1 = perimeter(new_file_name)
        rat = p1 / p
        print('Perim Poly', c1, '/ Perim Poly' , count , '=' , rat)
        file_name = new_file_name
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Give it 5 points, going in counter-clockwise order. It finds :

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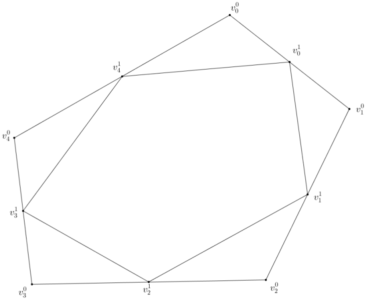
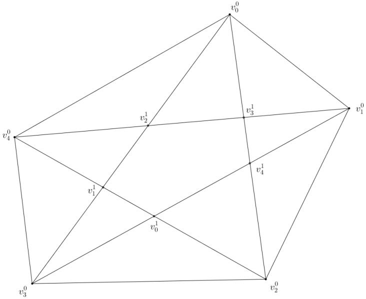
- Draws a picture of the n iterations
- Gives the vertices of Π^k

Two Types of Maps

Can we unify these two maps ?



The midpoint map



The pentagram map

The Generalized Pentagram Map (GPM)

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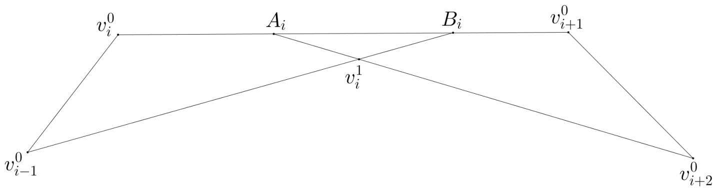

- Let Π be an n-gon with vertices $v_0^0, v_1^0, \ldots, v_{n-1}^0$.

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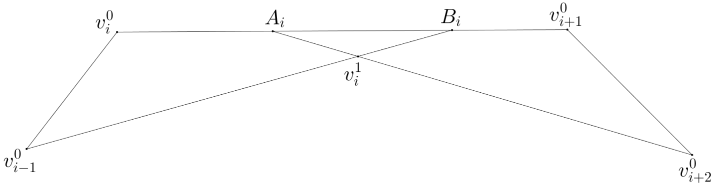

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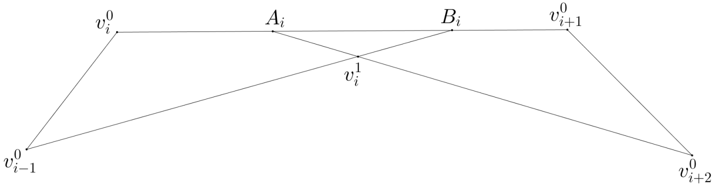

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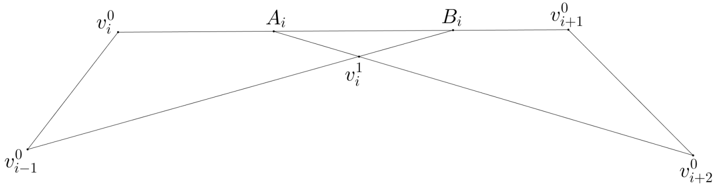

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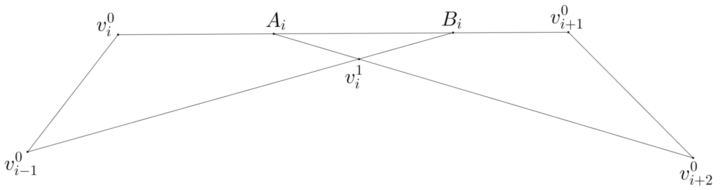

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- Call the intersection v_i^1.

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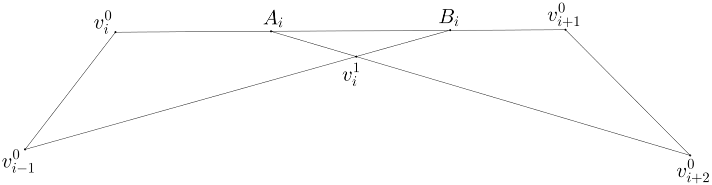

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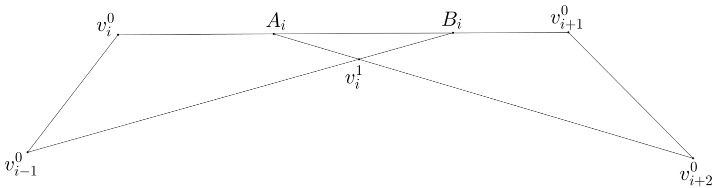
- Apply this process to each edge to form the vertices of an n-gon $T(\Pi)$.

The Generalized Pentagram Map (GPM)



- Let Π be an n-gon with vertices $v_0^0, v_1^0, \ldots, v_{n-1}^0$.
- Choose $a_i, b_i \in [0,1]$ with $a_i \leq b_i$ for $i = 0, 1, \ldots, n-1$.
- Construct points A_i, B_i such that $a_i = \frac{v_i^0 A_i}{v_i^0 v_{i+1}^0}$ and $b_i = \frac{v_i^0 B_i}{v_i^0 v_{i+1}^0}$.
- Connect v_{i-1}^0 to B_i and v_{i+2}^0 to A_i.
- Call the intersection v_i^1.
- Apply this process to each edge to form the vertices of an n-gon $T(\Pi)$.
- Applying this process k times gives us $T^k(\Pi)$.

Representing the Map

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Some Examples

$f(T) = (0, 0, \ldots, 0) \implies$

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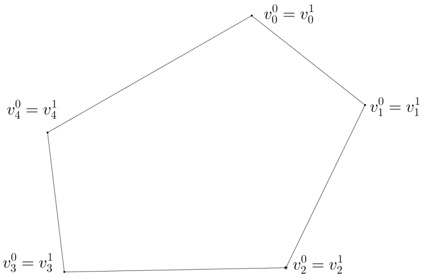

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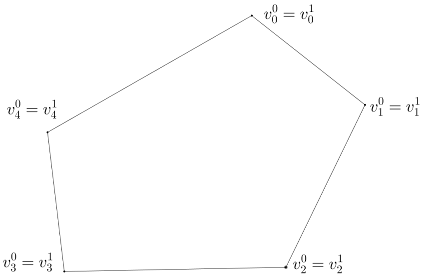

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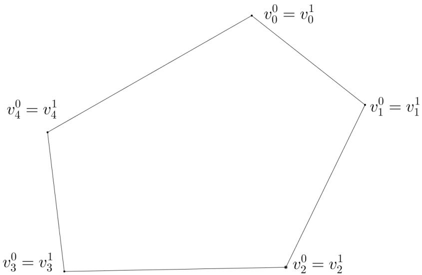
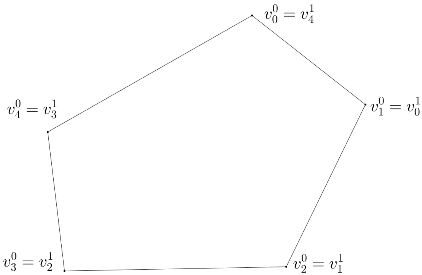


$f(T) = (1, 1, \ldots, 1) \implies T$ is a relabeling of the vertices.



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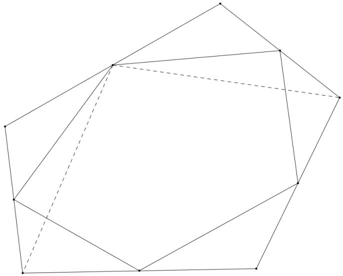

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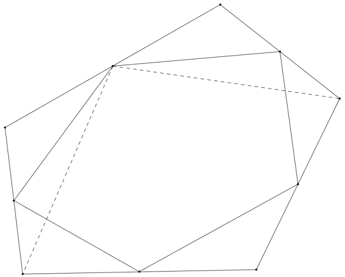

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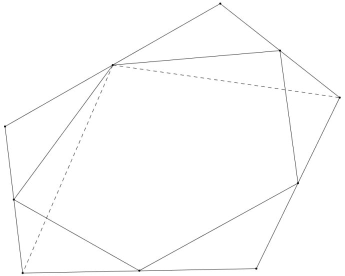
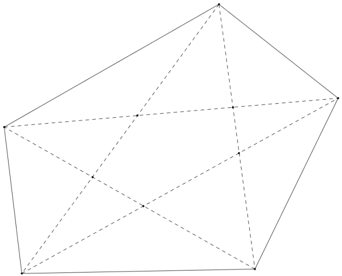
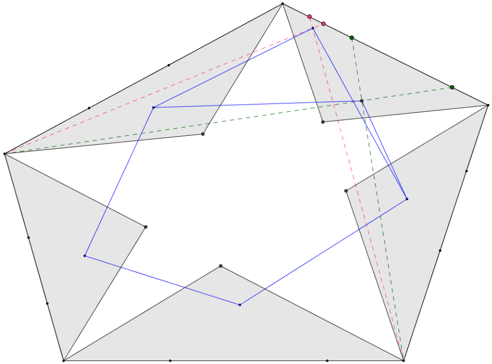


$f(T) = (0, 1, 0, 1, \ldots, 0, 1) \implies T$ is the pentagram map.



Intuition for the Map on Convex Polygons



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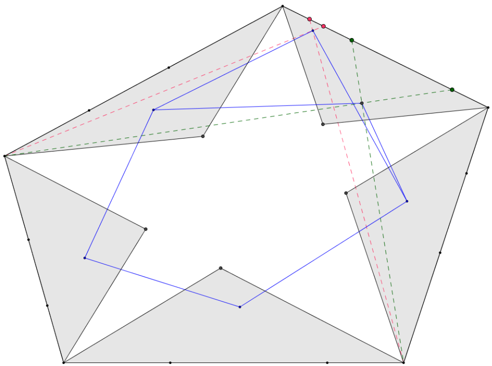

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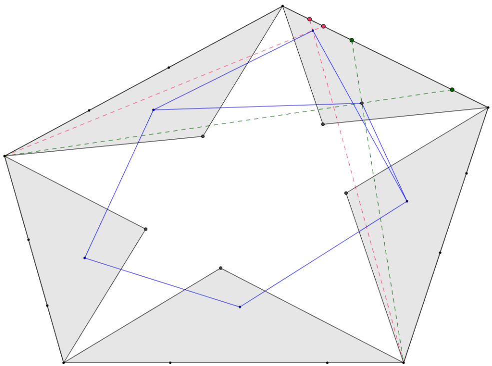

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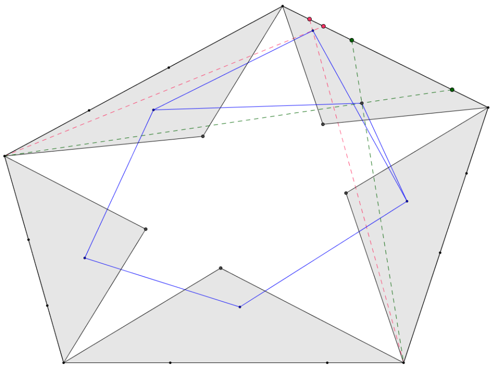

- Gray regions are the overlap of two consecutive vertex triangles.
- The vertices of $T(\Pi)$ lie inside separate gray regions.
- Each vertex of $T(\Pi)$ can lie anywhere in its corresponding region without affecting the configuration of the other vertices.

Convexity and the GPM

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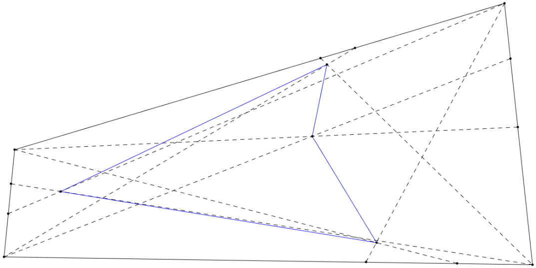
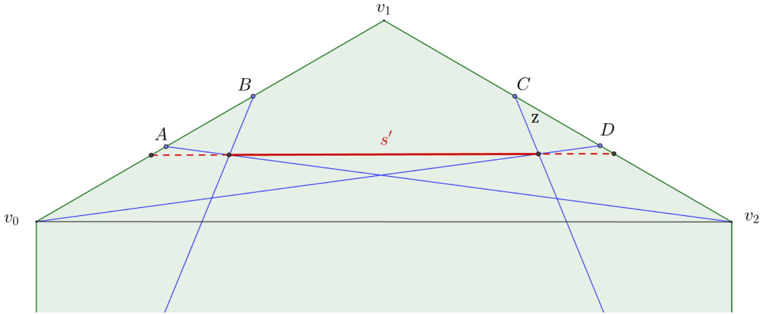
  - Unfortunately, no.



A Special Type of GPM

Let Π be a regular n-gon and let T be a GPM such that $f(T) = (m, 1 - m, m, 1 - m, \ldots, m, 1 - m)$ for some $m \in [0, \frac{1}{2}]$.



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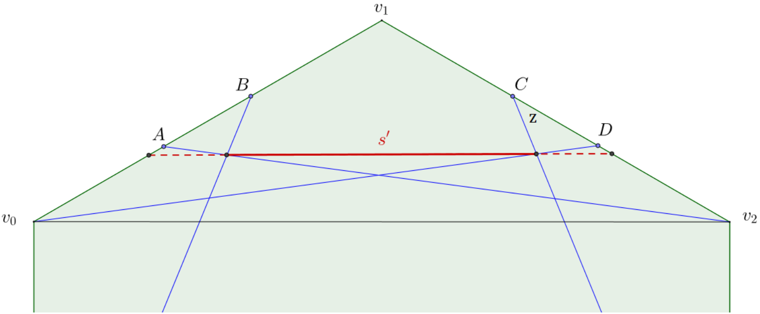

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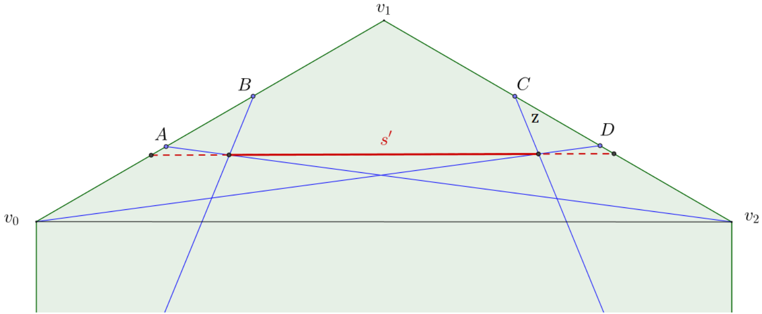

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Found using :

- Multiple Law of Sines applications
- Similar triangles
- Symmetry of the regular polygon

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- So T is a convexity preserving GPM on regular polygons and $T^k(\Pi)$ converges to a point.

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- This very "normal" type of GPM preserves regularity and decreases side length in a predictable way.

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Proposition

Let T_1 and T_2 be GPMs on a convex n-gon Π such that $f(T_1) = (a_0, b_0, \dots, a_{n-1}, b_{n-1})$ and $f(T_2) = (x, b_0, \dots, a_{n-1}, b_{n-1})$ where $a_0 \leq x$. Then $A(T_1(\Pi)) \leq A(T_2(\Pi))$.

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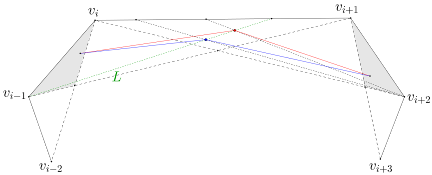

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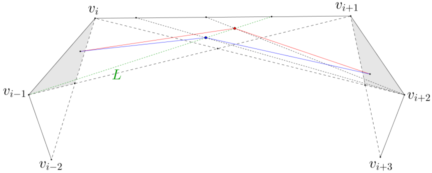
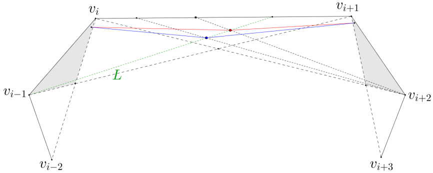


$T_2(\Pi)$ is not convex at vertex 0



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Corollary

Let T_P be the pentagram map and T be any other GPM on a convex polygon Π. Then $A(T_P(\Pi)) < A(T(\Pi))$.

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- The process seen in the previous proposition terminates with the pentagram map.
- Recall that last time we proved $\frac{A(T_P^{k+1}(\Pi))}{A(T_P^k(\Pi))} < \frac{14}{15}$ where Π is a pentagon.
- We can use this corollary to obtain a better bound on the rate of area reduction for the pentagram map on pentagons.

A Better Bound

The following proposition is due to Dan Ismailescu et al. [3].

Proposition

Let T_m be a GPM on a convex pentagon Π such that $f(T_m) = (m, m, \ldots, m)$. Then $\frac{A(T_m^{k+1}(\Pi))}{A(T_m^k(\Pi))} < 1 - m(1 - m)$.

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- By the proposition on the previous slide, $\frac{A(T_P^{k+1}(\Pi))}{A(T_P^k(\Pi))} < \frac{A(T_m(T_P^k(\Pi)))}{A(T_P^k(\Pi))} < 1 - m(1-m)$.

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- On the interval $[0, 1]$, the function $g(x) = 1 - x(1 - x)$ is attains a minimum of $\frac{3}{4}$ at $x = \frac{1}{2}$.
- So $\frac{A(T_P^{k+1}(\Pi))}{A(T_P^k(\Pi))} < \frac{3}{4} \implies \frac{A(T_P^k(\Pi))}{A(\Pi)} < \left(\frac{3}{4}\right)^k$.

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Proposition

Let Π be a convex pentagon and let T be a convexity preserving GPM on Π such that $f(T) = (a_0, b_0, a_1, b_1, \ldots, a_4, b_4)$ with $a_i \leq m \leq b_i$ for $i = 0, 1, \ldots, 4$. Then $T^k(\Pi)$ shrinks to a region of zero area. In particular,

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Proof :

- Apply the top proposition to each coordinate of $f(T)$.

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- We proved convergence to a point for a special type of GPM applied to regular polygons.
- We obtained a better bound of the rate of area decrease for the pentagram map.
- We proved that a restricted class of GPMs applied to a convex pentagon shrinks to a region of zero area. Furthermore, we provided a bound on the rate of area decrease.

Future GPM Directions

- Given a polygon Π, find a sufficient condition for a *GPM* to be a convexity-preserving map on Π.

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- Built upon previously established results by Richard Schwartz.
- Proved convergence to a point for a restricted class of pentagons.

THANK YOU

- Dr. Sun
- Professor Vollmar for his Python expertise
- Missouri State University for hosting us
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