Polychromatic Sums and Products in Finite Fields

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  - Arithmetic progressions (Roth, Green-Tao)

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- $AB = \{3, 6, 9, 10, 20, 30\}$

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- Elekes - $\frac{5}{4}$, Solymosi - $\frac{4}{3}$, Konyagin-Shkredov have the record with $\frac{4}{3} + c$ for some $c > 0$

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- Green-Tao proved that there are aribtrarily long arithmetic progressions of primes. Their theorem says, for every natural number, k, there exists arithmetic progressions of primes with k terms.

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- Open Problems In Partition Regularity (Hindman, Leader, Strauss), monochromatic $(x, y, x + y, xy)$ in $\mathbb{N}$.
- Monochromatic Sums and Products (Green, Sanders), monochromatic $(x, y, x + y, xy)$ in finite fields.

Polychromatic triples

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- This is different from the monochromatic triples and quadruples before, where all of the elements would all come from the same set, A_i.
- Note that this doesn't always happen. No polychromatic quadruples can exist in $\mathbb{Z}_{(4n)}$, where the color classes are $A_j = \{x \in \mathbb{Z}_{(4n)} : x \equiv j \pmod 4\}$.

Additive triples

- **Theorem 1:** If $k \geq 3$, for a large prime, p, then any k-coloring of $\mathbb{Z}_p$, where each color class has roughly the same size (either $\lceil \frac{p}{k} \rceil$ or $\lfloor \frac{p}{k} \rfloor$ elements), must admit a polychromatic triple of the form $(x, y, x + y)$.

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- When working in $\mathbb{Z}_q$, for q not necessarily prime, our results weaken.
- **Theorem 2:** There exists an additive polychromatic triple of the form $(x, y, x+y)$ in $\mathbb{Z}_q$ for k-coloring whenever we have $k > q^{\frac{1}{2}+\varepsilon}$, for every $\varepsilon > 0$.

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- **Corollary 1:** There exists a multiplicative polychromatic triple of the form (x, y, xy) in $\mathbb{Z}_p$ for k-coloring whenever we have $k > q^{\frac{1}{2}+\varepsilon}$, for every $\varepsilon > 0$.

Multiplicative triples

- A group is called **cyclic** if there exists an element, $g \in \mathbb{Z}_p^*$, called a **generator**, such that every element in the group can be written as g^j, for some $j \in \mathbb{N}$.

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- Therefore, the behavior of nonzero products in $\mathbb{Z}_p$ is isomorphic to the behavior of sums in $\mathbb{Z}_q$, where $q = (p - 1)$.
- So we apply Theorem 2 to the sets of exponents of g that correspond to each color class.

Notation

- We introduce the notation $A \widehat{\subseteq}_e B$, to mean that A is a subset of B, except for possibly a small exceptional set. That is to say, that A is **essentially** a subset of B. More precisely, for some small, specified constant,

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- **Lemma:** If p is a large prime, and A, B, and C are disjoint subsets of $\mathbb{Z}_p$, each of size n or $n+1$, with $\frac{p}{3}+1 > n > 10$, and possibly have the same size, then there exists a triple, $(x, y, x+y)$, where no two of the elements come from the same set.

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- We will prove the lemma by showing that we cannot have $A+B \subseteq A \cup B$ and $A+C \subseteq A \cup C$ simultaneously, which will mean that we have a polychromatic triple.

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- **Cauchy-Davenport Theorem:** For additive subsets of $\mathbb{Z}_p$, A and B: $|A+B| \geq \min\{|A|+|B|-1, p\}$.

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- **Vosper's Theorem:** If $|A+B| = |A|+|B|-1$ then A and B are arithmetic progressions with the same step size.
- **Hamidoune-Rødseth Theorem:** If $|A+B| = |A|+|B|$ then A and B are $\widehat{=}_1$ arithmetic progressions with the same step size.

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- $|A + B| = |A| + |B| - 1$ (Vosper)
  $|A + B| = |A| + |B|$ (Hamidoune-Rødseth)
- In either the case of Vosper's Theorem or the Hamidoune-Rødseth Theorem, we will have that our color classes must **essentially** be arithmetic progressions.

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- $A + B = \{a_0 + b_0 + su : s \in [0..(n+m-2)]\}$

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- The subscript of 1 follows from the fact that we are guaranteed that $A+B$ can be missing no more than one element from the set $\{a_0 + b_0 + su : s \in [0..(n+m-1)]\}$, by Cauchy-Davenport.

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- $C \widehat{=}_1 [c_0..(c_0 + l)]$
- Our sumsets are now of the following form $A + B \widehat{=}_1 \{a_0 + b_0 + s : s \in [0..(n + m - 1)]\}$ and $A + C \widehat{=}_1 \{a_0 + c_0 + s : s \in [0..(n + l - 1)]\}$.

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- By way of contradiction, we will assume that we do not have a polychromatic triple of the form $(x, y, x + y)$.

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- So we have that $(A + B) \subseteq (A \cup B)$ and $(A + C) \subseteq (A \cup C)$.

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- So $(A-a_0)$ is either the first or second half of $[b_0..(b_0+n+m)]$ and $(B-a_0)$ is the rest.

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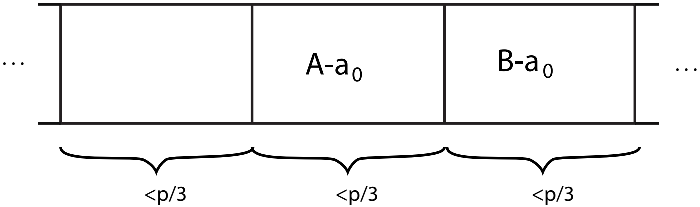

As each subset of $\mathbb{Z}_p$ is of size less than $p/3$, neither set can wrap all the way around to border both sides of the other. This figure ignores the possible exceptional elements.

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- Since $(A - a_0)\widehat{=}_1[0..n]$, we have that either

$$(i) \quad (B - a_0)\widehat{=}_4[b_0..(b_0 + m)] \text{ (left half)},$$

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- But this reasoning also applies with A and C, meaning that three disjoint sets of size n have to be contained in an interval of about $2n$ integers, with no more than 4 exceptional elements per set. This is a contradiction for $n > 12$.

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- Rearranging, we get that there exist n values of $(a_i - a_j)$, for a fixed i due to the n choices of j.
- Since there are n choices for a_i, the total number of elements that could be added to A to get A is $\leq n^2$.

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- So, if we violate this, then there must be a polychromatic triple for $k > q^{\frac{1}{2}+\varepsilon}$, for any $\varepsilon > 0$.

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- We can always find a polychromatic triple with more than four color classes
- We set the following restrictions on our sets and graph the corresponding equations:

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- $x + y \neq a_i, b_i$ for every $a_i \in A, b_i \in B$ and where i ranges from 0 to $(n - 1)$

Proof of Theorem 1 (part 2)

- We now count the number of choices of x and y that will **not** give a polychromatic triple. Using an inclusion-exclusion argument (illustrated on the next slide)with m as the number of elements in $A \cup B$ that x and y cannot be, we have $3p(m+1) - (2(m+1)^2 + m) + (1 + 3m + T) - (S_4) < p^2$

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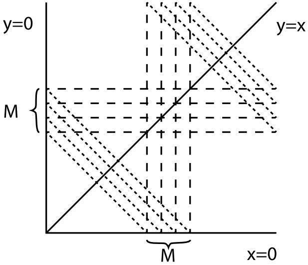
- So, $3p + 3pm - 2m^2 - 4m - 2 + T - S_4 < p^2$

Inclusion-exclusion figure



This is a graph of all of the points, (x, y), that will not yield a polychromatic triple. The full lines are $x = 0, y = 0$, and $y = x$. The vertical dashed lines are the cases of $x \in M$, where the horizontal dashed lines are the cases where $y \in M$. Finally, the dotted lines indicate points, (x, y), such that $(x + y) \in M$.

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- From this, we can compute $k \geq 4$ and $p > -\frac{k}{k-2}$.

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p	Color Class 1	Color Class 2	Color Class 3
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- As of yet, no further examples have been found when p is greater than 7.

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8	2, 3, 7	0, 4, 6	1, 5
9	1, 4, 8	0, 3, 6	2, 5, 7
10	3, 7, 8, 9	2, 4, 6	0, 1, 5
12	1, 4, 5, 7	2, 8, 10, 11	0, 3, 6, 9

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- No examples have been found for color classes in which no additive polychromatic triples occur in $\mathbb{Z}_q$ when $k = 3$.

Future work

- Generalize Theorem 2 for fewer sets. We currently have guaranteed the existence of a polychromatic triple in $\mathbb{Z}_q$ for k-colorings with $k > q^{\frac{1}{2}+\epsilon}$, for any $\epsilon > 0$. Can we also guarantee the existence of a polychromatic triple in $\mathbb{Z}_q$ for k-colorings with smaller k?

Future work

- Generalize Theorem 2 for fewer sets. We currently have guaranteed the existence of a polychromatic triple in $\mathbb{Z}_q$ for k-colorings with $k > q^{\frac{1}{2}+\epsilon}$, for any $\epsilon > 0$. Can we also guarantee the existence of a polychromatic triple in $\mathbb{Z}_q$ for k-colorings with smaller k?
- Computationally, polychromatic quadruples seem to exist rather often. How can we guarantee their existence?

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