Conjugacy Class Graphs of Dihedral and Permutation Groups

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Overview

What's a Group?

Definition

A group is a set G paired with a binary operation, $*$, that satisfies:

1. Closure: If $a, b \in G \implies a * b \in G$.
2. Associativity: $a, b, c \in G, (a * b) * c = a * (b * c)$
3. Identity: $\exists e \in G$ such that $e * g = g * e = g, \forall g \in G$.
4. Inverses: $\exists g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e, \forall g \in G$

If the operation is commutative, the group is a commutative, or abelian, group. Examples include $(\mathbb{Z}, +)$ integers under addition, D_{2n} the rotations and flips of an n-gon, and S_n the set of all permutations of n elements.

Conjugacy Classes

Definition

Let G be a group. Then, the conjugacy class of $a \in G$ is the set denoted as $cl(a)$, where $cl(a) = \{xax^{-1} \mid x \in G\}$.

If G is abelian, than each conjugacy class has size 1 because $xax^{-1} = xx^{-1}a = a$.

Conjugacy Classes of S_n

- Each permutation can be represented as disjoint cycles. Example: $(13)(12)(46)(45) = (123)(456)$
- The conjugacy class of a k cycle is the class containing all possible k cycles, and only k cycles. In other words, the cycle structure of every element in a conjugacy class is the same.

What's a Graph?

Definition

A graph, G, consists of a set of vertices, V, and a set of edges, E, which are defined by two vertices.

Definition

A complete graph, K_n, is a simple undirected graph in which every pair of n distinct vertices is connected by a unique edge.

Big Idea

Let G be a group. We create $\Gamma(G)$ by computing the conjugacy classes of $G - Z(G)$. A node is produced by every conjugacy class and labeled with the cardinality of the class, c_i. Lastly, an edge connects two vertices only if the $\gcd(c_{i_1}, c_{i_2}) > 1$. The main focus is to classify all graphs of $\Gamma(D_{2n} \times D_{2m})$ and to study the completeness of $\Gamma(S_n)$.

Big Idea

Let
$a_{1,1}, a_{1,2} \dots a_{1,k_1}$ be the conjugacy classes of G_1,
$a_{2,1}, a_{2,2} \dots a_{2,k_2}$ be the conjugacy classes of G_2,
$\vdots$
$a_{m,1}, a_{m,2} a_{1,k_m}$ be the conjugacy classes of G_m
The size of the conjugacy classes of $G_1 \times G_2 \times \dots G_m$ are all products in the form of $a_{1,j_1} \cdot a_{2,j_2} \cdots a_{m,j_m}$ where $1 \leq j_1 \leq k_1$, $1 \leq j_2 \leq k_2, \dots 1 \leq j_m \leq k_m$.

General Theorems

Definition

An induced subgraph of a graph is another graph, formed from a subset of the vertices of the graph and all of the edges connecting pairs of vertices in that subset.

Lemma

Let $\mathcal{G} = \Gamma(G_1 \times G_2 \times \cdots \times G_k)$ be a graph. Then $\Gamma(G_i)$ is an induced subgraph of $\mathcal{G}$ for $1 \leq i \leq k$.

Proof.

This is a consequence of how the size of the conjugacy class of $G_1 \times G_2 \times \cdots \times G_k$ are computed. Since we multiply every conjugacy class size by the size of $|\text{cl}(\text{id})|$, there is an exact copy of $\Gamma(G_i)$ for $1 \leq i \leq k$. Thus $\Gamma(G_i)$ is a subgraph $\mathcal{G}$. □

Examples

To compute $\Gamma(S_3 \times S_3)$, since S_3 has conjugacy class sizes 1,2, and 3 so since $\gcd(2,3) = 1$ S_3 isn't complete, and neither is $\Gamma(S_3 \times S_3)$.

The size of each conjugacy class of S_5 are :

1, 10, 15, 20, 20, 24 and 30 and

the size of each conjugacy class of S_7 are:

1,, 21, 70, 105, 105, 210, 210, 280, 420, 420, 504, 504, 630, 720, and 840.

Since $\gcd(10,21) = 1$ $\Gamma(S_7 \times S_5)$ is not complete.

General Theorems

Theorem

If at least one of the graphs $\Gamma(G_1), \Gamma(G_2) \ldots \Gamma(G_k)$ *are connected, then* $\Gamma(G_1 \times G_2 \cdots \times G_k)$ *is connected.*

General Theorems

Theorem

Let $\mathcal{G} = \Gamma(G_1 \times G_2 \cdots \times G_k)$ be a complete graph, then $\Gamma(G_1), \Gamma(G_2), \ldots \Gamma(G_k)$ are complete.

Again, the converse is not true. Let's look at $\Gamma(S_5)$ and $\Gamma(S_7)$, both of which are complete. We show that $\Gamma(S_5 \times S_7)$ is not complete because $|\operatorname{cl}((12))|$ of S_5 is 10 and $|\operatorname{cl}((12))|$ of S_7 is 21.

Graphs of Dihedral Groups

$\Gamma(D_{2n})$

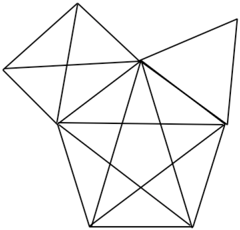
In general, the three possibilities for the graph of D_{2n} are:

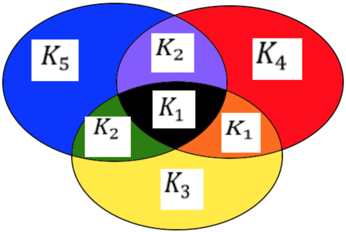
$$\Gamma(D_{2n}) = \begin{cases} K_{\frac{n-1}{2}} \bigsqcup K_1 & n \equiv 1 \bmod 2 \\ K_{\frac{n}{2}-1} \bigsqcup K_2 & n \equiv 2 \bmod 4 \\ K_{\frac{n}{2}+1} & n \equiv 0 \bmod 4 \end{cases}$$

Notation

The join of a K_a and a K_b along a K_c will be denoted as $K_{\{a,b;c\}}$
The join of a K_a and K_b along a K_d, K_a and K_c along a K_e, K_b and K_c along a K_f, and the K_d, K_e, and K_f along a K_g will be denoted $K_{\{a,b,c;d,e,f;g\}}$.

Example





This would be a $K_{\{5,4,3;2,2,1;1\}}$

Classification of $\Gamma(D_{2n} \times D_{2m})$

1. The cases where $n \equiv 1 \bmod 2$ and $m \equiv 2 \bmod 4$
   1. $K_{\{\frac{mn+4m+6n-11}{4},\ n+3,\ \frac{m+6}{2};\ n-1,\ \frac{m-2}{3},\ 2;\ 0\}}$ when $\gcd(\frac{m}{2}, n) = 1$
   2. $K_{\{\frac{mn+4m+6n-11}{4},\ \frac{m+8}{2};\ \frac{m+2n-4}{2}\}}$ when $\gcd(\frac{m}{2}, n) > 1$
2. The cases where $n \equiv 0 \bmod 4$ and $m \equiv 1 \bmod 2$
   1. $K_{\{\frac{mn+6m+3n+2}{4},\ \frac{n+6}{2};\ \frac{n+2}{2}\}}$ when $\gcd(m, \frac{n}{2}) = 1$
   2. $K_{\{\frac{mn+6m+3n+2}{4},\ \frac{2m+n+8}{2};\ \frac{2m+n+4}{2}\}}$ when $\gcd(m, \frac{n}{2}) > 1$
3. The cases where $n \equiv 0 \bmod 4$ and $m \equiv 2 \bmod 4$
   1. $K_{\{\frac{mn+6m+6n+4}{4},\ n+6;\ n+2\}}$ when $\gcd(\frac{m}{2}, \frac{n}{2}) = 1$
   2. $K_{\frac{mn+6m+6n+4}{4},\ m+n+8;\ m+n+4\}}$ when $\gcd(\frac{m}{2}, \frac{n}{2}) > 1$

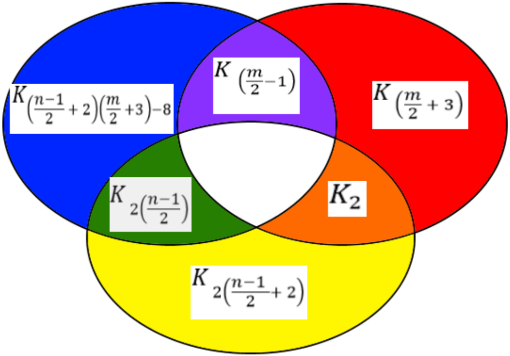
$\Gamma(D_{10} \times D_{12})$

$D_{10} \times D_{12}$ where $n \equiv 1 \bmod 2$, $m \equiv 2 \bmod 4$.
In this case, $n = 5$ and $m = 6$. Using the generalizations for $\Gamma(D_{2n})$, we know $\Gamma(D_{10})$ produces $K_2 \bigsqcup *$ and $\Gamma(D_{12})$ produces $K_2 \bigsqcup K_2$. We can obtain the graphs of this type with two subcases: $\gcd(n, \frac{m}{2})=1$, $\gcd(n, \frac{m}{2}) > 1$. When the $\gcd(n, \frac{m}{2}) > 1$, the graph can be generated using

$$K_{\{\left(\frac{n-1}{2}+2\right)\left(\frac{m}{2}+3\right)-8,\left(\frac{m}{2}+3\right)+2\left(\frac{n-1}{2}+2\right)-2;\left(\frac{m}{2}-1\right)+2\left(\frac{n-1}{2}\right)\}}$$

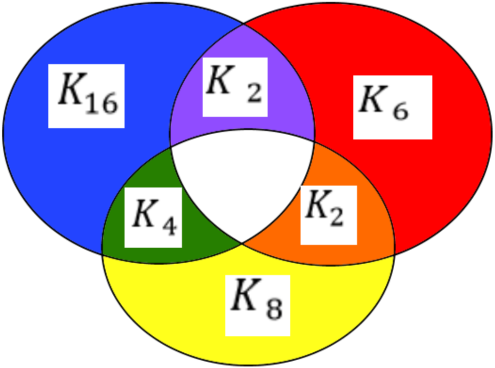
$\Gamma(D_{10} \times D_{12})$

However, when the gcd$(n, \frac{m}{2})$=1, the graph is obtained using



$\Gamma(D_{10} \times D_{12})$

Since the gcd(5, $\frac{6}{2}$)=1, we use the second formula to show $\Gamma(D_{10} \times D_{12})$ will form



Classification of $\Gamma(D_{2n} \times D_{2m})$

1. The cases where $n, m \equiv 1 \bmod 2$
   1. $K_{\{\frac{mn+3m+3n-8}{4},\ \frac{m+3}{2},\ \frac{n+3}{2};\ \frac{m-1}{2},\ \frac{n-1}{2},\ 1;\ 0\}}$ when $\gcd(m, n) = 1$
   2. $K_{\{\frac{mn+3m+3n-8}{4},\ \frac{m+n+4}{2};\ \frac{m+n-2}{2}\}}$ when $\gcd(m, n) > 1$
2. The case where $n, m \equiv 0 \bmod 4$
   $K_{\frac{mn+6m+6n+20}{4}}$
3. The case where $n, m \equiv 2 \bmod 4$
   1. $K_{\{\frac{mn+6m+6n-12}{4},\ m+6,\ n+6;\ m-2,\ n-2,\ 4;\ 0\}}$ when $\gcd(\frac{m}{2}, \frac{n}{2}) = 1$
   2. $K_{\{\frac{mn+6m+6n-12}{4},\ m+n+8;\ m+n-4\}}$ when $\gcd(\frac{m}{2}, \frac{n}{2}) > 1$

$\Gamma(D_{10} \times D_{14})$

$D_{10} \times D_{14}$ where $n, m \equiv 1 \bmod 2$
In this case $n = 5$ and $m = 7$. Since $n \equiv 1 \bmod 2$, we use

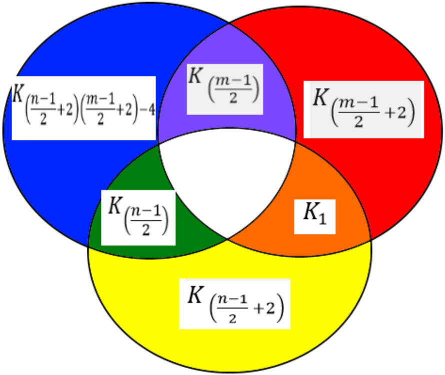
$$K_{\frac{n-1}{2}} \bigsqcup *$$

to produce their graphs individually. Using the general formula, $\Gamma(D_{10})$ produces $K_2 \bigsqcup *$ and $\Gamma(D_{14})$ produces $K_3 \bigsqcup *$. We generalize how to produce the graph of their graph with two subcases: $\gcd(n, m){>}1$, $\gcd(n, m) = 1$. When $\gcd(n, m) > 1$, we generalize the graph of their product to be

$$K_{\{\left(\frac{n-1}{2}+2\right)\left(\frac{m-1}{2}+2\right)-4,\left(\frac{n-1}{2}+2\right)\left(\frac{m-1}{2}+2\right)-1;\left(\frac{m-1}{2}\right)+\left(\frac{n-1}{2}\right)\}}$$

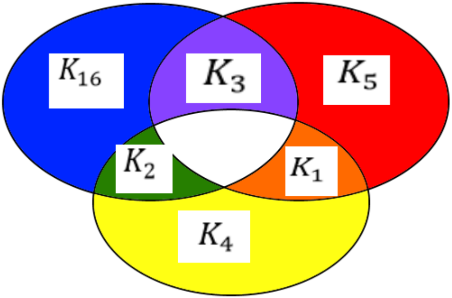
$\Gamma(D_{10} \times D_{14})$

When the $\gcd(n, m) = 1$, we generalize the graph of their product to be



$\Gamma(D_{10} \times D_{14})$

In $D_{10} \times D_{14}$, the gcd $(5, 7) = 1$, therefore we use the second equation to obtain our graph.
The $\Gamma(D_{10} \times D_{14})$ will produce



Graphs of Permutation Groups

Permutation Groups

Definition

An integer partition of x is a way of writing x as a sum of positive integers. We denote the number of partitions there are of x by $n(x)$.

Permutation Groups

Theorem

For every prime $p \geq 5$, S_p generates a complete graph.

To prove this, we need the following lemma:

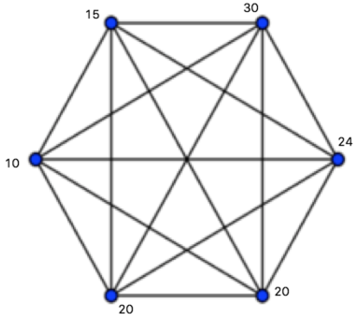
Lemma

The smallest, nontrivial conjugacy class graphs of S_n is the class of a single transposition, for $n \geq 7$.

For $n < 7$ the smallest class is still the class of transpositions, but there are other classes just as small.

$\Gamma(S_5) = K_6$

The size of each conjugacy class is: 1, 10, 15, 20, 20, 24 and 30.



$\Gamma(S_p)$ is Complete

The cardinality of each conjugacy class in S_p can be expressed as:

$$\frac{p!}{\prod_{i=1}^{p}(n_i)! \cdot i^{n_i}}$$

which is how we were able to show all the conjugacy classes had a factor in common, except for the conjugacy class of p-cycles. We then used the lemma to show that the smallest conjugacy classes (the transpositions) were

$$\binom{p}{2} = \frac{p(p-1)}{2} > p.$$

Permutation Groups

There is a direct correlation to the cycle structures of S_p and the partitions of n, there are exactly $n(p)$ many conjugacy classes of S_p, where $n(x)$ is the number of partitions of a positive integer x. Thus, since the conjugacy class of the identity does not receive a vertex, $\Gamma(S_p) = K_{n(p)-1}$.

$\Gamma(S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k})$

Now we study when $\Gamma(S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k})$ generates a complete graph. In an earlier example, we noted that $\Gamma(S_5 \times S_7)$ is not complete.
However, both $\Gamma(S_5)$ and $\Gamma(S_6)$ are complete, and $\Gamma(S_5 \times S_6)$ is as well. In fact, $\Gamma(S_5 \times S_n)$ iff $\Gamma(S_6 \times S_n)$.

$\Gamma(S_5 \times S_n)$ and $\Gamma(S_6 \times S_n)$

As mentioned earlier, the size of each conjugacy class of S_5 are :
1, 10, 15, 20, 20, 24 and 30
The size of each conjugacy class of S_6 are:
1, 15, 15, 40, 45, 90, 90, 120, 120, 144

$\Gamma(S_5 \times S_n)$ and $\Gamma(S_6 \times S_n)$

The class of S_5 with 10 elements corresponds to the primes 2 and 5. Likewise, the class with 15 elements corresponds to the primes 3 and 5, and the class with 24 elements corresponds to the primes 2 and 3.

For S_6, the class with 40 elements corresponds to the primes 2 and 5. So if the class of 10 elements in S_5 is connected to a vertex, v, so is the class with 40 elements. Similarly the class with 45 elements is going to connect to every vertex the class of 15 elements (in S_5) connects to. And the class with 144 elements is going to connect to all the vertices the class of 24 elements connects to.

$\Gamma(S_{p^2} \times S_p)$ is complete

Theorem

The graph $\Gamma(S_{p^2} \times S_p)$ is complete for all prime, $p \geq 5$.

$\Gamma(S_{p^2} \times S_p)$ is complete

First, we showed that the only class, $\text{cl}(a)$ that's size didn't have a factor of p, of S_{p^2}, is the class of p many p cycles. For example, the only class of S_9 whose size is not divisible by 3 is $\text{cl}((123)(456)(789))$.

Next we showed that no conjugacy class had a size relatively prime to $|\text{cl}(a)|$. We already know that no class of S_p has size exactly p, and we used a very similar argument to prove the same for S_{p^2}, using the transposition lemma.

Conclusion

- Found general theorems about the completeness and connectedness of products of graphs.
- Classified the conjugacy class graphs of the form of $\Gamma(D_{2n} \times D_{2m})$.
- Found some conditions on when $\Gamma(S_k \times S_l)$ is complete.

Future Work

- Further the classification of when $\Gamma(S_n \times S_m)$ is complete.
- When is $\Gamma(S_{n_1} \times S_{n_2} \times \cdots \times S_{n_k})$ complete?
- What are the possibilities of $\Gamma(D_{2m} \times S_n)$?

The End