Hausdorff Distance and Hausdorff Dimension

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Introduction

First approach:

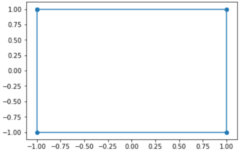


Figure: unit square

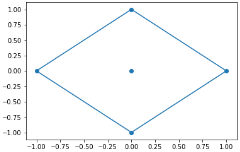


Figure: 1st iteration

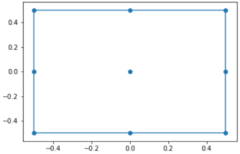


Figure: 2nd iteration

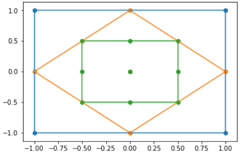


Figure: First two iterations with the original unit square

⋮

Second approach:

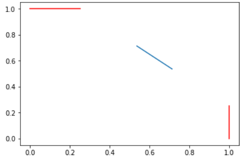


Figure: recursion1

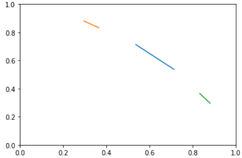


Figure: recursion2

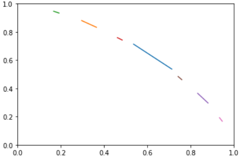


Figure: recursion3

⋮

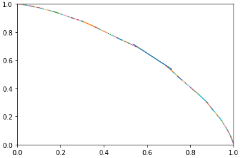


Figure: recursion curve

The total length of the segments

$$= \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} a$$

$i = 1 : a$
$i = 2 : 2\frac{a}{3^1}$
$i = 3 : 4\frac{a}{3^2}$
$\vdots$
as $i \to \infty$: $a \sum_{i=1}^{\infty} \left(\frac{2}{3}\right)^{i-1} = a\frac{1}{1-\frac{2}{3}} = 3a$

Hausdorff Distance

Definition

Hausdorff distance between sets A and B in K (where K is the collection of all nonempty compact subsets of a metric space) as:

$$r(x, B) = inf\{d(x, b) : b \in B\}$$

$$p(A, B) = sup\{r(a, B) : a \in A\}$$

$$h(A, B) = max\{p(A, B), p(B, A)\}$$

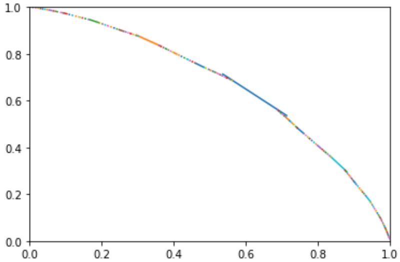


Figure: recursion curve

Hausdorff Dimension

Definition

In a general metric space (X, p), let U be any subset of X. We define the diameter of U as:

$$diam(U) = sup\{p(x, y) : x, y \in U\}$$

Definition

Let S be any subset of X. The Hausdorff Outer Measure Dimension of d bounded by δ, where $\delta > 0$:

$$\mathcal{H}^d_\delta(S) := inf\left\{\sum_{i=1}^{\infty}(diamU_i)^d : S \subseteq \bigcup_{i=1}^{\infty} U_i, diam(U_i) < \delta\right\}$$

We conclude that

$$\lim_{\delta \to 0+} \mathcal{H}^d_\delta(S) = \mathcal{H}^d(S)$$

exist, but may be infinite.

Theorem

If $\mathcal{H}^p(A) < \infty$, then $\mathcal{H}^q(A) = 0$ for all $q > p$. Also if $\mathcal{H}^p(A) > 0$, then $\mathcal{H}^q(A) = \infty$ for all $q < p$.

Definition

Hausdorff Dimension:

$$dim_H(A) = p$$

iff $H^p(A)$ is finite, nonzero number.

- Remark: If the set A has Hausdorff dimension $dim_H(A) = k$, then

$$\mathcal{H}^p = \begin{cases} 0, & \text{if } k < p \\ finite, nonzero\ number, & \text{if } k = p \\ \infty, & \text{if } k > p \end{cases}$$

Example: Koch Curve

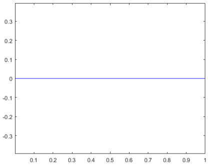


Figure: A

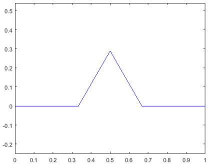


Figure: $S(A)$

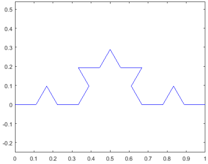


Figure: $S^2(A)$

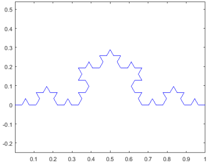


Figure: $S^3(A)$

$\vdots$

- Remark: If the set A has Hausdorff dimension $dim_H(A) = k$, then

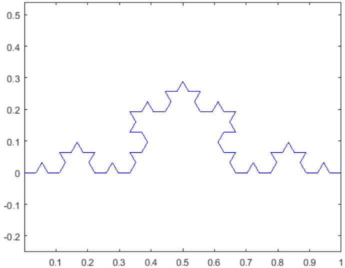
$$\mathcal{H}^p = \begin{cases} 0, & \text{if } k < p \\ finite, nonzero\ number, & \text{if } k = p \\ \infty, & \text{if } k > p \end{cases}$$

- $$\mathcal{H}^1(K) = \lim_{n \to \infty} (\frac{4}{3})^n = \infty$$ $$dim_H(K) > 1$$

- $$\mathcal{H}^2(K) = 0$$ $$dim_H(K) < 2$$

- $$1 < dim_H(K) < 2$$

- $$\lim_{i\to\infty}[\mathcal{H}^p(S^{i+1}(A)) - \mathcal{H}^p(S^i(A))] = 0$$



- $S^i(A) = (\frac{1}{3})^i$, numbers of lines in $S^i(A)$ is 4^i

- $$\mathcal{H}^p(S^i(A)) = 4^i\left(\frac{1}{3}\right)^{ip}, i = 1, 2, 3, 4, \ldots$$

- $$\lim_{i\to\infty}[4^{i+1}\left(\frac{1}{3}\right)^{(i+1)p} - 4^i\left(\frac{1}{3}\right)^{ip}] = 0$$

$$\lim_{i\to\infty} 4^i\left(\frac{1}{3}\right)^{ip}(4\left(\frac{1}{3}\right)^p - 1) = 0$$

$$\mathcal{H}^p(S^i(A))(4\left(\frac{1}{3}\right)^p - 1) = 0$$

Since $\mathcal{H}^p(S^i(A)) > 0$

$$4\left(\frac{1}{3}\right)^p - 1 = 0$$

$$p = \frac{log\frac{1}{4}}{log\frac{1}{3}} \approx 1.26$$

In our case:

$$S^i(A) = \frac{a}{3^{i-1}}, i = 1, 2, 3, 4, \ldots$$

Numbers of lines in $S^i(A)$ is 2^{i-1}

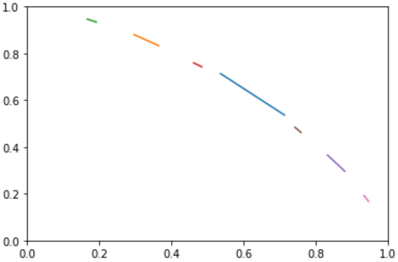


Figure: recursion3

Thus,

$$\mathcal{H}^p(S^i(A)) = 2^{i-1}a\left(\frac{1}{3}\right)^{(i-1)p}$$

$$\lim_{i\to\infty}[2^i a\left(\frac{1}{3}\right)^{(i)p} - 2^{i-1}a\left(\frac{1}{3}\right)^{(i-1)p}] = 0$$

$$\lim_{i\to\infty} 2^{i-1}a\left(\frac{1}{3}\right)^{(i-1)p}(2a\left(\frac{1}{3}\right)^{p} - 1) = 0$$

$$\mathcal{H}^p(S^i(A))(2\left(\frac{1}{3}\right)^{p} - 1) = 0$$

Hence, $2\left(\frac{1}{3}\right)^p - 1 = 0 \Rightarrow p = \frac{log\frac{1}{2}}{log\frac{1}{3}} \approx 0.631$

Future Works

- Find the unique value of a such that the curve constructed is closed.
- Conjecture: The limit curve is closed if and only if the sequence constructed is Cauchy.