Generalized Rainbow Configurations

Mike Desgrottes [1], David Soukup[1], Renjun Zhu[1]

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Background

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    - Classical Ramsey Theory - Schur's Theorem:
      Coloring the natural numbers into finite number of colors, then there must a color class contain a triple with $(x, y, x + y)$
- **Goal**: find a condition on the coloring to guarantee a rainbow configuration.

What's a Rainbow Configuration?

Suppose we have a set X={0 , 1 , **2** , 3 , **4** , 5 }
Q: Can you find $(x, y, x + y)$ such that x, y, and x+y are different colors?

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Hint:

- 1 + 2 = 3
- 2 + 3 = 5
- 0 + 1 = 1
- 2 +2 = 4
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- 0 + 1 = 1
- 2 +2 = 4
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We also see (1, 4 , 5) is a triple with all distinct color elements.

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The **configuration space** is the set $\mathcal{F} = \{(x_1, x_2, \ldots, x_k) \in X^k\}$ for $k \geq 3$

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A **rainbow tuple** is an element $f \in \mathcal{F}$ such that the elements $\pi_1(f), \pi_2(f) \ldots \pi_k(f)$ all belong to distinct color classes. If no rainbow tuples exist, then our configuration space $\{X, \mathcal{F}, c\}$ is **rainbow-free**.

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- From our previous example, $X = \{0, 1, \mathbf{2}, 3, \mathbf{4}, 5\}$. The color classes are $\{2, 4\}, \{0, 5\}, \{1, 3\}$.
- Our configuration space $\mathcal{F} = \{(x, y, x + y) \in X^3\}$
- Our coloring is **not** rainbow-free since there are rainbow triples, $(1, 4, 5)$, $(2, 3, 5)$, *etc*.

Known Results

Theorem: Ceja, Cook, and Hayden (2016)

If we let $X = \mathbb{Z}/p$, for large prime p, and $k \geq 3$, then for every partitioning of X into k colors with each of size $\lceil \frac{p}{k} \rceil$ or $\lfloor \frac{p}{k} \rfloor$, there must exist a rainbow tuple of the form $(x_1, x_2, x_1 + x_2)$.

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Theorem: Schönheim (1990)

For every 3-coloring of $1, 2, 3, \ldots n$, with every color class with at least $n/4$ elements, there exists a rainbow tuple $(x_1, x_2, x_1 + x_2)$.

Theorem: Senger (2017)

Fix $c > 0$. Given a coloring of $\mathbb{F}_q^2$, with q sufficiently large, where no color class has size $\geq cq^2$, if unit equilateral triangles exist in $\mathbb{F}_q^2$ then there must be a rainbow unit equilateral triangle.

Constant M

Definition

Given $\mathcal{F} \subset X^k$, we define $M_{i,j}$ as the constant such that for any $x = (x_1, x_2, \ldots, x_k) \in \mathcal{F}$, we have $|\{y \in \mathcal{F} : y_i = x_i, y_j = x_j\}| \leq M_{i,j}$

$$M = \sum_{i<j} M_{i,j}$$

1. Given $\mathcal{F} = \{(x, y, x + y) \in X^3\}$, $M = M_{1,2} + M_{1,3} + M_{2,3} = 3$
   - If we fix any two coordinates, then we can uniquely determine the other coordinate. So, there's only one tuple in $\mathcal{F}$, namely $M_{1,2} = M_{2,3} = M_{1,3} = 1$

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2. Given $\mathcal{F} = \{(x, y, x+y, xy) \in X^4\}$, $M = M_{1,2} + M_{1,3} + M_{1,4} + M_{2,3} + M_{2,4} + M_{3,4} = 5 \cdot 1 + 2 = 7$

Finite Case Theorem(FCT)

Theorem

Let X be a finite set of size n, and $\mathcal{F} \subset X^k$ be a set of k-tuples. If no color class has size $\geq Cn$, where $|\mathcal{F}| \geq Dn^2$, and $C < \frac{2D}{9M}$, then there must be a rainbow k-tuple in $\mathcal{F}$.

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3. Use $|\mathcal{F}|$ bounds to get a lower bound on the number of color classes.
4. We use what we know about the color classes' size to reach a contradiction.

Merging

Definition

Define a coloring λ to be a **merging** of λ' if x, y are the same color under λ', then they also have the same color under λ.

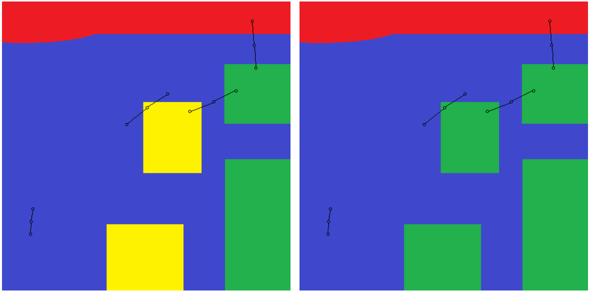

Figure: Merge color classes can destroy but not create rainbow configurations.

Proof.

- We merge the color classes until every color class has size between $\frac{1}{2}Cn$ and $\frac{3}{2}Cn$. Let s denote the number of color class after this process.

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- $|\mathcal{F}| \leq M\sum_{i=1}^{s} n_i^2 \leq M\sum_{i=1}^{s} \left(\frac{3}{2}Cn\right)^2 \leq s\frac{9M}{4}C^2n^2$

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  it implies $s \geq \frac{4}{9M}\frac{1}{C^2}\frac{|\mathcal{F}|}{n^2} \geq \frac{4D}{9MC^2}$
- Finally, X has at least $\left(\frac{1}{2}Cn\right) \cdot \frac{4D}{9MC^2} > n$ elements, for $C < \frac{2D}{9M}$
  Contradiction!

□

Finite Case Corollary

Corollary 1

Suppose we have a coloring of a finite abelian group, G. If all color class has size $< \frac{2}{27}|G|$ then there must be a rainbow triple of the form $(x, y, x + y)$.

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- thus, $D = 1$, and $C < \frac{2D}{9M} = \frac{2}{27}$

Corollary 2

If we color $\mathbb{F}_q$ such that each color class has size $< \frac{(2-o(1))}{63}q$, then there must be a rainbow quadruple of the form $(x, y, x+y, xy)$.

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Corollary 3

If we color a finite (additive) abelian group G, with no (nonidentity) element of order $< k$, then if there are no rainbow k-arithmetic progressions, at least one color class has size $\geq \frac{2}{9\binom{k}{2}}|G|$.

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Corollary 4

If we color $[1..n]$ such that there are no rainbow triples of the form $(x, y, x+y)$, then at least one color class has size $\geq \frac{1-o(1)}{27}n$.

Pathological Coloring

Example

Suppose we have the following color classes on $\mathbb{Q}$, where $a, b \in \mathbb{Z}$ for $b \neq 0$

- color class 0: $\{\mathbb{Z}\} \bigcup \{\frac{a}{b} \in \mathbb{Q} : 2 \nmid a, b\}$

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  ...
- color class i: $\{\frac{a}{2^i b} : 2 \nmid a, b\}$

If $x \in$ color class i, and $y \in$ color class j, then $x + y \in$ color class $\max(i, j)$. Here we have infinitely many classes, but still **rainbow-free** of the form $(x, y, x + y)$.

Pathological Coloring

Example

Suppose we color vector space $\mathbb{R}$ over $\mathbb{Q}$ as following, and let $\{x_i\}_{i\in I}$ be a well-ordered basis.

- Define A_j to be the set of linear combinations $\sum_{i\in I} c_i x_i$ with all but fintely many c_i equal to zero and $j =$min$(i : c_i \neq 0)$

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- The color classes are A_j. We see that if $a_i \in A_i$ and $a_j \in A_j$, then $a_i + a_j, a_i - a_j, a_j - a_i \in A_{min(i,j)}$

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Here we have uncountably many color classes, but again it's a **rainbow-free** of the form $(x, y, x+y)$ in this coloring.

What is more do we need?

Measure!!!

Before we talk about the size of the set X, we can do that earlier because our sets were finite. However, now we can have base space and color classes be **infinite**. For that, we need a way to talk about the "size" of an infinite case.

Measure Space to the Rescue

Measure Space

1. X: finite or infinite set
2. Σ: **Measurable sets**
   - a collection of subsets of X, also known as σ-algebra, which is closed under its compliment, countable unions, and intersections;
   - contains X, and $\varnothing$.
3. μ: **Measure**, a function map $\Sigma \to \mathbb{R}^+ \bigcup \{0, \infty\}$ such that:
   - $\mu(\varnothing) = 0$;
   - $\mu(\bigcup E_i) = \Sigma \mu(E_i)$, where $E_i \bigcap E_j = \varnothing$

(X, Σ, μ) is called measure space.

X is Finite: $\mu(X) < \infty$

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Definition

Let (X, μ) be a measure space, a coloring of X is **tractable** if every color class is a measurable subset of X, and there are only at most countably many color classes.

Infinite Case Theorem (ICT)

Let (X, μ) be a **finite measure space**, $n = \mu(X) < \infty$, and $\mathcal{F} \subset X^k$ be a set of k-tuples. If we have a **tractable coloring** where no color class has measure $\geq Cn$, with $m(\mathcal{F}) \geq D\binom{k}{2}n^2$, and $C < \frac{2D}{9M}$, then there must be a rainbow k-tuple in $\mathcal{F}$.

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$$m(\mathcal{F}) = \sum_{i<j} m_{ij}(\mathcal{F})$$

where

$$m_{ij}(\mathcal{F}) = \iint |\{y \in \mathcal{F} : x_i = y_i, x_j = y_j\}| dx_i dx_j$$

Note: The finite case is considered μ as counting measure.

Merging

Remark

Suppose we have an original coloring, λ_0. Let Λ be the set of all coloring that are merging of λ_0 and have no color classes of size $> Cn$.

Here, apply Zorn's lemma so that we can talk about maximum and minimum measure of the color class.

Sketch of ICT Proof

Proof.

1. Similar to the finite case, we start with assuming that our coloring is rainbow-free.
2. Merge color classes until the measure of each the color class, $\mu(A)$, is between $\frac{1}{2}Cn$ and $\frac{3}{2}Cn$
3. We obtain contradiction based on the number of coloring.

□

Corollary 1

Fix a probability measure μ on the unit circle in the complex plane. If we color the circle with at least 14 equally sized $\mu-$measurable color classes, there must be a rainbow triple of the form (x, y, xy)

- We note that the measure space is finite since $\mu(X) = 1$.
- $\mathcal{F} = \{(x, y, xy) \in X^3 | x, y \in X\}$. We note that any two pairs of elements uniquely determine a triple in $\mathcal{F}$. So $m(\mathcal{F}) = 3$. Hence, $D = 1$.

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- We apply ICT with $D = 1$, $M = 3$, and $C \leq \frac{2D}{3M} = \frac{2}{27}$. We satisfy the color class because each color classes has $\mu-$measure $1/14 < 2/27$.

Corollary 2

If we split any square into 104 equally sized Lebesgue measurable color classes, then it must contain three points x, y, z such that $|x - y| = |z - y| = |x - z|$ with the points being distinct colors.

Definition

Let X have a coloring, and $x \in X^k$. Then x is **w-subrainbow** if x has no w components of the same color.

$(x_1, x_2, x_3, x_4, x_5)$ is 3-subrainbow, but $(x_1, x_2, x_3, x_4, x_5)$ is not.

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Definition

Let (X, μ) be a finite measure space, $\mathcal{F} \subset X^k$, and fix a $t \in [2..k-1]$. For any subset $S \subset [1..k]$, such that $|S| = t$, so write $S = \{s_1, s_2, \ldots, s_t\}$. Define **t-bound** of $\mathcal{F}$ be M_t, so $M_t = \sum_s M(S)$, where $M(S) = \sup_{(y_1,\ldots,y_k)\in\mathcal{F}} |\{x \in \mathcal{F} : x_{s_j} = y_{s_j}\}|$

$x = (x_1, x_2, x_3, \ldots, x_t, \ldots, x_k)$
$y = (y_1, y_2, y_3, \ldots, y_t, \ldots, y_k)$
Similarly,

$$m = \sum_S m(S)$$

where

$$m(S) = \int \cdots \int |\{y \in \mathcal{F} : x_{s_i} = y_{s_i}, i \in [1..t]\}| ds_1 \cdots ds_t$$

Theorem(Generalized)

Let (X, μ) be a **finite measure space**, and define $n = \mu(X)$. Let $\mathcal{F} \subset X^k$ be measurable, and let the **t-bound** be M, and suppose $m(\mathcal{F}) \geq D\binom{k}{t}n^t$ and $2 \leq w \leq t < k$. Then for any **tractable coloring** of X, if $\mu(A) < Cn$ for all color classes A. Then $\mathcal{F}$ must contain a w-subrainbow element, as long as $C^{w-1} < \dfrac{2^{w-1}D}{3^w M\binom{t}{w}}$

Proof.

We use the same technique as before □

Corollary

Let G be an abelian group, $2, 3 \nmid |G|$, with some coloring. Then:

1. If no color class has size $\geq \frac{1}{135}|G|$, there must be a **rainbow** quintuple of the form $(x, y, z, x + y + z, x + 2y + 3z)$

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Let G be an abelian group, $2, 3 \nmid |G|$, with some coloring. Then:

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Let G be an abelian group, $2, 3 \nmid |G|$, with some coloring. Then:

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Any Questions ??

Thank You ☺