Bounds on Subframes

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This set of lines forms a frame in a natural way; Jasper, Mixon, and Fickus use frames to give applications of this problem to coding theory in [3].

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In this context the study of frames has powerful applications to signal processing, wavelets, and data compression (see [1]).

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- $\mathbb{R}^d$ or $\mathbb{C}^d$ with the standard dot product $\langle v, w \rangle = v \bullet w = \sum v_i \overline{w_i}$.
- $L^2[0,1]$, the functions $f : [0,1] \to \mathbb{F}$ such that $\int_0^1 |f(x)|^2 dx$ converges, with $\langle f, g \rangle = \int_0^1 f(x) \overline{g(x)} dx$.

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Both of the examples of inner product spaces from last slide are also Hilbert spaces.

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1. The elements of F span $\mathcal{H}$.
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For example, if $\{\varphi_i\}$ happen to form an orthonormal basis, then $\sum_{i \in I} |\langle x, \varphi_i \rangle|^2 = ||x||^2$ so this is a frame with $A = B = 1$.

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A tight frame is **Parseval** if $A_F = B_F = 1$.

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So $A_F = B_F = 3/2$, meaning the Mercedes-Benz frame is tight ($\Omega(F) = 1$).

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So $A_F = B_F = 3/2$, meaning the Mercedes-Benz frame is tight ($\Omega(F) = 1$). It's not Parseval; we could make it Parseval by scaling the φ_i by a factor of $\sqrt{2/3}$.

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In short, the obscurity measures **how far from symmetry** our frame is (in a certain sense).

Question

Given a large frame, under what conditions does there exist a smaller frame of specified size with small obscurity?

Our strategy

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The last step relies on the following lemma.

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$$\Omega(F \cup G) \leq \max(\Omega(F), \Omega(G))$$

In other words, if we glue two frames together, the resultant frame is no worse than the frames we started with.

Proof of lemma

To see this, notice that $t_{F \cup G} = t_F + t_G$.

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This completes the proof of the lemma; using this we can build frames out of smaller ones.

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Using this lemma, we can prove the following result:

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Suppose we have a frame F consisting of n nonzero vectors in $\mathbb{R}^d$, with a the ratio between the largest and smallest norm, and $d|k \leq n$. Then if we can find d disjoint subsets $E_1, E_2, \ldots E_d \subset F$ each containing $\geq k/d$ vectors such that the angle between vectors in different subsets is $\geq \beta$, there is a subframe $E \subset F$ such that $|E| = k$ and:

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If $d > 2$ we need a stronger condition on F to get a good bound on A_E.

A word on angles

Incidentally, the **angle** between two vectors x, y is defined as the unique $\theta \in [0, \pi/2]$ such that

$$\cos\theta = \frac{|\langle x, y\rangle|}{||x||||y||}$$

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The reason for this nonstandard definition is that replacing a vector φ_i in a frame by $-\varphi_i$ does not affect obscurity since $|\langle x, \varphi_i\rangle|^2 = |\langle x, -\varphi_i\rangle|^2$. So in effect we want to measure the angle between *lines* and not *vectors*.

A_E in higher dimensions

In higher dimensions we need the following (stronger) assumptions to get the following (stronger) result:

Theorem

Suppose we have a frame F consisting of n nonzero unit vectors in $\mathbb{R}^d$, and $d|k \leq n$. Suppose further that there exists an orthonormal basis $\{f_i\}$ and an angle $\gamma < \pi/4$ such that for each f_i there exist at least k/d vectors of F with angle $\leq \gamma$ from f_i. Then there exists a subframe E of F with

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This means $\Omega(E) \leq \frac{1+\gamma}{1-\gamma}$.

A probabilistic bound

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Theorem

Suppose we pick N points uniformly at random on the unit circle, and $2 \ll r \ll N$ with $2|r$. Then for any $k > 2$, $2|k$, there is a $\geq q$ probability of finding a subframe F with $|F| = k$ and

$$\Omega(F) \leq \text{ctn}^2(\pi/4 - \pi/r) = \tan^2(\pi/4 + \pi/r)$$

so long as

$$\Phi^*(2k) \leq \sqrt{\frac{2(1-q)}{r}}$$

where Φ^* is the cdf of a normal distribution with mean N and standard deviation $\sqrt{N(r-1)}$.

Unpacking this theorem

Equivalently:

$$\frac{1}{\sqrt{2\pi N(r-1)}}\int_{-\infty}^{2k} e^{-\frac{(t-N)^2}{2N(r-1)}}\,dt \le \sqrt{\frac{2(1-q)}{r}}.$$

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Results in higher dimensions, though, would require bounds on sphere packings which are still open problems!

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