Non-Nilpotent Graphs of Groups

Andrew Davis, Julie Kent, Emily McGovern

Missouri State University

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Overview

Group

A group is an algebraic structure consisting of a set G and a binary operation $*$, with the following properties:

- Closure: If $a, b \in G$, then $a * b \in G$.
- Associativity: For all $a, b, c \in G$, $a * (b * c) = (a * b) * c$.
- Identity: There exists an element $e \in G$ where $a * e = e * a = a$ for all $a \in G$.
- Inverses: For all $a \in G$, there exists an element $a^{-1} \in G$, where $a * a^{-1} = a^{-1} * a = e$.

Group properties

- A group G is called Abelian if $a * b = b * a$ for all $a, b \in G$.
- A group G is said to be generated by x and y if every element of the group can be expressed in terms of x and y. We denote this group as $\langle x, y \rangle$.

More group properties

- The center of a group, denoted $Z(G)$, is the set of elements that commute with everything in the group.
- The centralizer of an element in a group $C_G(x)$ is the set of all elements that commute with that element.
- A group is called AC if the centralizer of every non-central element is an Abelian subgroup.

Nilpotent Groups

Commutator

For two sets A and B, the commutator of A and B is $[A, B] = \langle aba^{-1}b^{-1} |$ for all $a \in A, b \in b \rangle$

Now, for a group G, call $G^{'} = [G, G]$, $G^{''} = [G, G^{'}]$ and $G^{(n)} = [G, G^{(n-1)}]$

Nilpotent Group

A group G is called nilpotent if $G^{(n)} = \{1\}$ for some natural number n, where $\{1\}$ is the trivial group consisting of only the identity.

Remark: For an Abelian group, $aba^{-1}b^{-1} = baa^{-1}b^{-1} = 1$ for all $a, b \in G$ so $G^{'} = \{1\}$. Therefore, all Abelian groups are also nilpotent.

More on Nilpotent Groups

- All groups of order p^α, where p is prime, are nilpotent.

Weakly Nilpotent

A group G is weakly nilpotent if $\langle x, y \rangle$ is nilpotent for every $x, y \in G$

Nilpotentizers of Elements and Groups

For an element x in a group G, the nilpotentizer of x in G is
$nil_G(x) = \{g | g \in G \text{ and } \langle x, g \rangle \text{ is nilpotent}\}$.
For a group G, the nilpotentizer of G is
$nil(G) = \{g | g \in G, \langle g, h \rangle \text{ is nilpotent for all } h \in G\}$.

More on Nilpotent Groups

$\mathfrak{n}$-groups and $\mathfrak{nn}$-groups

A group G is a $\mathfrak{n}$-group if $nil_G(x)$ is a subgroup for all $x \in G$, and an $\mathfrak{nn}$-group if $nil_G(x)$ is a nilpotent subgroup for all $x \in G \backslash nil(G)$

Graphs

Graph

A graph G is defined by a set of vertices V and a set of edges E where the elements of E are unordered pairs of the elements of V. Two vertices $a, b \in V$ are connected if and only if $(a, b) \in E$.

Complete Multipartite Graph

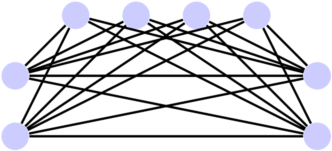
A graph is complete k-partite if its vertices can be partitioned into k sets, where vertices in the same set are not connected to each other, but all vertices in different sets are.

These graphs are denoted $K_{n_1,n_2,\ldots,n_k}$, where the n_i's are the size of the vertex sets.

If graph has j sets of the same cardinality, we denote the graph as $K_{n_1,n_2,\ldots,[n_k]^j}$.

More Graph Properties

The following graph is complete multi-partite ($K_{4,[2]^2}$):



Eulerian Graph

A graph Γ is called eulerian, if it includes a circuit that uses each edge exactly once. A graph is path-eulerian if it has a trail that uses each edge exactly once.

Theorem

A connected graph is eulerian if and only if all vertices have even degree, and graph is path-eulerian if and only if all but two vertices have even degree.

Non-nilpotent Graphs

Non-nilpotent Graph of a Group

For a group G, define the non-nilpotent graph $n_G(G)$ as the following:

- $V(n_G(G)) = G \setminus nil(G)$
- $(g, h) \in E(n_G(G))$ if and only if $\langle g, h \rangle$ is not nilpotent.

Dihedral Groups

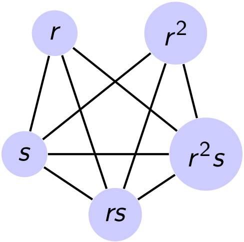
Dihedral groups represent the symmetries of an n-gon and are generated by a rotation r and a reflection s.

Dihedral Group

The dihedral group $D_{2n} = \langle r, s | r^n = 1, s^2 = 1, srs^{-1} = r^{-1} \rangle$.

Non-Nilpotent Graph of D_6

$V(n_G(D_6)) = \{r, r^2, s, rs, r^2s\}$



$n_G(D_6) \simeq K_{2,1,1,1}$

Structure of Non-Nilpotent Graphs of Dihedral Groups

Theorem

For a dihedral group $D_{2^k m}$, where m is odd, $n_G(D_{2^k m}) \simeq K_{2^{k-1}(m-1),[2^{k-1}]^m}$

Justification:

- $nil(D_{2^k m}) = \langle r^m \rangle$, $|nil(D_{2^k m})| = 2^{k-1}$, $|V(n_G(D_{2^k m})| = 2^k m - 2^{k-1} = |V(K_{2^{k-1}(m-1),[2^{k-1}]^m})|$
- Any two remaining powers of r generate a cyclic group, so all $2^{k-1}(m-1)$ are disjoint.
- The remaining elements are grouped into sets of the following form $\{r^i s, r^{m+i} s, r^{2m+i} s, ..., r^{2^{k-1} m+i} s\}$. Each group has 2^{k-1} elements and there are m of these groups.

Properties of Non-Nilpotent Graphs of Dihedral Groups

Corollary

The chromatic number of $n_G(D_{2^k m})$ is $m+1$.

Justification: $K_{2^{k-1}(m-1),[2^{k-1}]^m}$ has $m+1$ disjoint vertex sets, and each requires a different color.

Corollary

The girth (length of shortest cycle) of $n_G(D_{2^k m})$ is 3.

Corollary

The diameter (longest minimum path between any two vertices) of $n_G(D_{2^k m})$ is 2.

Universal Vertices

Definition

A universal vertex of a graph G is a vertex that is adjacent to every other vertex in G.

Note that if y is a universal vertex in a non-nilpotent graph, then $nil_G(y) = nil(G) \cup y$.

Theorem

For a finite non-nilpotent group G, if $n_G(G)$ has a universal vertex, then $nil(G) = \{1\}$. Furthermore, if there is at least one universal vertex in $n_G(G)$, there are exactly $\frac{|G|}{2}$ universal vertices in $n_G(G)$.

Universal Vertices

Justification:

- By [4,4.1], $|nil(G)| \mid |nil_G(x)| - |nil(G)|$ for all $x \in G \backslash nil(G)$.
- Let y be a universal vertex. Then $|nil_G(y)| = |nil(G)| + |\{y\}|$. Thus $|nil(G)| = 1$.
- $|nil(G)| = 1 \Rightarrow nil(G) = 1 \Rightarrow Z(G) = 1$
- Then $|C_G(y)| = 2$, so the order of the conjugacy class of y is $\frac{|G|}{2}$
- Conjugation conserves nilpotency, so all elements in the conjugacy class of y are universal vertices of $n_G(G)$.
- No element x outside of the conjugacy class of y can be universal vertices, because then the conjugacy class of x would also have order $\frac{|G|}{2}$, and every element in the group would have order 2.

Frobenius Groups

Definition

A subgroup $H \leq G$ is malnormal if $xHx^{-1} \cap H = 1$ for all $x \in G - H$.

Definition

A group G is a Frobenius group if it has a proper, nontrivial malnormal subgroup H. The Frobenius kernel K is a subgroup defined as $G - \cup xHx^{-1}$ for all $x \in G$. G can be represented as $K \rtimes H$.

A result on Frobenius groups states that if H is even, then K is abelian.

Universal Vertices Continued

Theorem

$n_G(G)$ contains universal vertices if and only if $G = K \rtimes C_2$, K an abelian group of odd order and $tht^{-1} = h^{-1}, t \in C_2, h \in K$.

Universal Vertices Continued

Justification:

- Let $G = K \rtimes C_2$, K an abelian group of odd order and $tht^{-1} = h^{-1}, t \in C_2, h \in K$.
- Since $|K|$ is odd, $\langle t, h \rangle \simeq D_{2|h|} \simeq \langle t, ht \rangle$, which is non-nilpotent.
- Then $nil(t) = \langle t \rangle$, and t is a universal vertex of $n_G(G)$.
- Now we need to prove the converse.

Universal Vertices Continued

Justification:

- Let G be a finite non-nilpotent group so that $n_G(G)$ contains a universal vertex t. Note that t must have order 2.
- $\langle t \rangle$ is a proper malnormal subgroup of G, because if $\{xtx^{-1}\} \cap \{t\}$ intersected non-trivially, then x would commute with t, and t wouldn't be a universal vertex. Therefore G is a Frobenius group.
- Then $G = K \rtimes \langle t \rangle$. Since $\langle t \rangle$ is even, K is abelian.
- By the previous theorem, there must be $|K|$ universal vertices that are the coset of K. Let tK be the left coset of K. Then tk*tk=1, so $tkt^{-1} = k^{-1}$.
- This also means that $|K|$ must be odd, or else there would be some element of order 2 in K and $tkt^{-1} = k$, a contradiction.

Universal Vertices Continued

Corollary

$n_G(K \rtimes C_2) \simeq K_{m-1,[1]^m}$, K an abelian group of odd order m, and $tht^{-1} = h^{-1}, t \in C_2, h \in K$.

Justification: $n_G(K \rtimes C_2)$ contains m universal vertices by previous results. $nil(K \rtimes C_2) = \{1\}$, and K is abelian, so $K - \{1\}$ will be a disjoint set of vertices of size m-1. This means $n_G(K \rtimes C_2) \simeq K_{m-1,^m1}$

Direct Products of Groups

Direct Product of Groups

For two groups G and H, the direct product of G and H, denoted $G \times H$, is the set $\{(g, h) | g \in G, h \in H\}$ with the operation defined as $(g_1, h_1) * (g_2, h_2) = (g_1 *_G g_2, h_1 *_H h_2)$.

Co-normal Products of Graphs

Co-normal Product of Graphs

For two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$, the co-normal product of Γ_1 and Γ_2, denoted $\Gamma_1 \times \Gamma_2$, is defined as:

- $V(\Gamma_1 \times \Gamma_2) = V_1 \times V_2$
- $((a_1, b_1), (a_2, b_2)) \in E(\Gamma_1 \times \Gamma_2)$ if $(a_1, a_2) \in E_1$ or $(b_1, b_2) \in E_2$.

Non-nilpotent graphs of direct products

Lemma

If G and H are groups, then:

1. If elements g_i and g_j are connected in $n_G(G)$, then (g_i, h_i) and (g_j, h_j) are connected in $n_G(G \times H)$ for all $h_i, h_j \in H$. Likewise, if elements h_i and h_j are connected in $n_G(H)$, then (g_i, h_i) and (g_j, h_j) are connected in $n_G(G \times H)$ for all $g_i, g_j \in G$.
2. If g_i and g_j are not connected in $n_G(G)$ and h_i and h_j are not connected in $n_G(H)$, then (g_i, h_i) and (g_j, h_j) are not connected in $n_G(G \times H)$

Non-nilpotent graphs of direct products

Justification:

- The homomorphic image of a nilpotent group is nilpotent. The image of $\langle (g_1, h_1), (g_2, h_2) \rangle$ under the projection map π_G is $\langle g_1, g_2 \rangle$ (and similarly for π_H).
- If $\langle g_1, g_2 \rangle$ or $\langle h_1, h_2 \rangle$ is non-nilpotent, then $\langle (g_1, h_1), (g_2, h_2) \rangle$ is non nilpotent.
- We know $\langle (g_1, h_1), (g_2, h_2) \rangle$ is a subgroup of $\langle g_1, g_2 \rangle \times \langle h_1, h_2 \rangle$.
- If $\langle g_1, g_2 \rangle$ and $\langle h_1, h_2 \rangle$ are nilpotent, then so is $\langle (g_1, h_1), (g_2, h_2) \rangle$.

Corollary

If A is nilpotent, then two vertices (a_1, g_1) and (a_2, g_2) are connected in $n_G(A \times G)$ if and only if g_1 and g_2 are connected in $n_G(G)$.

Direct Product of Nilpotent Groups

Theorem

Let G be a group and A be an nilpotent group. Then $n_G(A \times G) \simeq I_{|A|} \times n_G(G)$, where $I_{|A|}$ is the graph with $|A|$ isolated vertices.

Justification:

- $|V(n_G(A \times G))| = |A \times G| = |A| * |G|$ and $|V(I_{|A|} \times n_G(G))| = |V(I_{|A|}) \times V(n_G(G))| = |A| * |G|$
- In $I_{|A|} \times n_G(G)$, by definition, two vertices are connected only if they are connected in $n_G(G)$.
- This is exactly the condition given by the previous corollary for two vertices to be connected in $n_G(A \times G)$.

Isomorphic Non-Nilpotent Graphs

Theorem [4, 4.1]

Let G and H be two groups such that $n_G(G) \simeq n_G(H)$. If G is finite and non-nilpotent so is H. Moreover $|nil(H)|$ divides

$$gcd(|G| - |nil(G)|, |G| - |nil_G(x)|, |nil_G(x)| - |nil(G)| : x \in G \setminus nil(G))$$

The authors of [4] show that if $n_G(G) \simeq n_G(H)$, then $|G| = |H|$ when:

- G is a group of order pq, where p and q are prime
- $G = D_{2n}$ for n not a power of 2.

Isomorphic Non-Nilpotent Graphs

Theorem

If $G = N_n \times D_{2^k m}$, where N_n is a nilpotent group of order n and there exists H such that $n_G(G) \simeq n_G(H)$, then $|G| = |H|$

Justification:

- We know $|G| - |nil(G)| = |H| - |nil(H)|$, must show $|nil(G)| = |nil(H)|$
- By the previous theorem, $y = |nil(H)|$ divides $2^{k-1}n$.
- Since H is an $\mathfrak{nn}$-group, $|nil_H(x)| = 2^{k-1}(m-1)n + y$ divides $|H| = (2^k m - 2^{k-1})n + y$.
- This only holds when $y = 2^{k-1}n$, so $|nil(G)| = |nil(H)|$.

Isomorphic Non-Nilpotent Graphs

- We have also shown that if $n_G(G) = n_G(H)$ and $G = N_n \times A_4$, then $|G| = |H|$.
- Conjecture: This statement also holds for $G = N_n \times H$, where H is a group of order pq.
- Goal: use these results to prove or provide a counterexample for the general problem.

Semi-Direct Product of Groups

Semi-Direct Product of Groups

For groups K and H, the semi-direct product, denoted $K \rtimes H$, of these groups is the group of all ordered pairs $\{(k,h)|k \in K, h \in H,\}$, with the operation $(k_1,h_1) * (k_2,h_2) = (k_1\phi(h_1)(k_2), h_1h_2)$, where $\phi : H \to Aut(K)$ is a homomorphism.

Semi-Direct Product of Cyclic Groups

For cyclic groups C_m and C_n,
$C_m \rtimes C_n = \langle a, b | a^m = b^n = 1 \text{ and } bab^{-1} = a^k \rangle$, where $k^n \equiv 1 \mod m$.

Non-Nilpotent Graphs of Semi-Direct Products

Theorem

For p and q prime, $n_G(C_p \rtimes C_q) \simeq K_{(p-1),[q-1]^p}$, where $p|q-1$.

Justification:

- All subgroups are either isomorphic to C_p or C_q. By definition, only one C_p (powers of a).
- We can see $|b| = |ab| = q$, but $b \neq (ab)^k$, so there are p disjoint copies of C_q, by Sylow's Theorems.
- Any two items from the same subgroup will generate that group, but any two from different groups will generate the whole group.
- So the graph has the desired structure.

Non-Nilpotent Graphs of Semi-Direct Products

We have also shown:

- For p, q, r prime, if $r \not| p - 1$, $n_G(C_{pq} \rtimes C_r) \simeq n_G((C_q \rtimes C_r) \times C_p)$.
- Otherwise, $n_G(C_{pq} \rtimes C_r) \simeq K_{pq-1,[r-1]^{pq}}$
- We know $n_G(C_{15} \rtimes C_4)$ is not complete multipartite, so we cannot extend this pattern further.

Non-commuting Graphs

Non-commuting graph of a group

For a group G, define the non-commuting graph $\Gamma(G)$ as the following:

- $V(\Gamma(G)) = G \setminus Z(G)$
- $(g, h) \in E(\Gamma(G))$ if and only if $\langle g, h \rangle$ is not Abelian.

- By Nongsaing and Saikia, $n_G(G) = \Gamma(G)$ if G is centerless and AC.
- Goal: find a relaxed condition

A relaxed condition for equality

Theorem

If G is a group, then the non-commuting graph ($\Gamma(G)$) and the non-nilpotent graph ($n_G(G)$) are equal if and only if all nilpotent subgroups are Abelian.

Justification:

- Under this condition, $nil(G) = Z(G)$ and so $V(n_G(G)) = V(\Gamma(G))$.
- If two elements are connected in $n_G(G)$, they are by definition connected in $\Gamma(G)$.
- Two elements connected in $\Gamma(G)$ generate a non-Abelian group, which under our condition is non-nilpotent, so they are connected in $n_G(G)$.
- Conversely, if $\Gamma(G) = n_G(G)$ and there exists a non-Abelian nilpotent subgroup S, we have that there exists a pair of vertices in S that generate a non-Abelian nilpotent subgroup, which is a contradiction.

Relationship between the conditions

Theorem

If a group G is centerless and AC, then all nilpotent subgroups of G are Abelian.

Justification:

- By Abdollahi and Zarrin, centerless AC means all nilpotentizers are Abelian.
- If a subgroup S is non-Abelian and nilpotent, then it contains $\langle a, b \rangle$ which is also non-Abelian and nilpotent for some values of a, b.
- But $\langle a, b \rangle$ is a subgroup of $nil_G(a)$, and is therefore Abelian, which is a contradiction. So S is Abelian.

Remark

D_{12} has all Abelian nilpotent subgroups, but has center $\{r^3, 1\}$ and is not centerless AC. So the converse does not hold.

Conditions for Complete Multipartite

Theorem (Nongsiang and Saikia)

The non-nilpotent graph of a group G is complete multi-partite if and only if G is an $\mathfrak{nn}$ group.

Justification:

- $x \in nil_G(y), y \in nil_G(z)$.Then $z \in nil_G(y)$. As the nilpotentizer of y is weakly nilpotent, $x \in nil_G(z)$.
- Then $x \not\sim y$, $y \not\sim z$ in $n_G(G)$, so $x \not\sim z$
- Similarly, the converse holds

Non-CMP non-nilpotent graphs

Some groups have non-nilpotent graphs that are not complete multi-partite. Some examples are:

- S_4 - the permutations of a set with 4 elements.
- $D_6 \times D_6$

$D_6 \times D_6$ is part of a larger class of groups whose non-nilpotent graphs are not complete multi-partite

Non-CMP non-nilpotent graphs

Theorem

If G,H are non-nilpotent groups, then $n_G(G \times H)$ is not complete multi-partite.

Justification:

- Assume $x, y \in G, \langle x, y \rangle$ is non-nilpotent. Let $h \in H$.
- $\langle (x, 1), (1, h) \rangle$ is abelian, so it is nilpotent and they aren't connected in the graph.
- The same is true for (1,h) and (y,1).
- However, $\langle (x, 1), (y, 1) \rangle$ is non-nilpotent, so they are connected.
- Then, by our definition of complete multi-partite, this can't be complete multi-partite.

Eulerian and Path-Eulerian

Definition

$\Gamma(G)$ is eulerian if $|G - C_G(x)|$ is even $\forall x \in G$, and is path-eulerian if $|G - C_G(x)|$ is even for all but two $x \in G$

Definition

$n_G(G)$ is eulerian if $|G - nil_G(x)|$ is even $\forall x \in G$, and is path-eulerian if $|G - nil_G(x)|$ is even for all but two $x \in G$

Eulerian Equivalence

Lemma

For any $x \in G$, $|x| \mid |nil_G(x)|$. (This was previously proven for $C_G(x)$ in [3])

Justification:

- $\langle x, y\rangle = \langle x, x^i y\rangle$ for all $i \in \mathbb{Z}$, so $y \in nil_G(x)$ if and only if $x^i y \in nil_G(x)$.
- $nil_G(x)$ is a disjoint union of sets of the form $\{x^i y | i = 0, \ldots, |x| - 1\}$ each having $|x|$ elements and $|x|$ divides $|nil_G(x)|$.

Eulerian Equivalence

Definition

For any subset $S \subseteq G$, let $n_2(S) = |\{x \in S | x^2 = 1\}|$

Lemma

For any $x \in G, |nil_G(x)| \equiv n_2(nil_G(x)) \bmod 2$.

Justification:

- $\langle x, y \rangle = \langle x, y^{-1} \rangle$, so the elements of $nil_G(x)$ occur in pairs $\{y, y^{-1}\}$ except when $y^2 = 1$.
- Then, letting p denote the number of pairs, $|nil_G(x)| = 2p + n_2(nil_G(x))$ and the result follows.

Eulerian Equivalence

Lemma

If two elements in a nilpotent group have relatively prime order, then they commute.

Justification:

- Since the group is nilpotent, it is the direct product of its Sylow subgroups.
- As $gcd(x, y) = 1$, they must be in different products of Sylow subgroups of G. Therefore, x and y commute.

Eulerian Equivalence

Theorem

For any $x \in G, |nil_G(x)| \equiv |C_G(x)| \mod 2$.

Justification:

- If $|C_G(x)|$ is odd, then $n_2(C_G(x)) = 1$, and $|x|$ is odd.
- Then any element of order two that generates a nilpotent subgroup with x commutes with x, so $n_2(C_G(x)) = n_2(nil_G(x))$ and $|nil_G(x)|$ is odd.
- If $|C_G(x)|$ is even and $|x|$ is even, then $|nil_G(x)|$ is even.

Eulerian Equivalence

- Let $|C_G(x)|$ be even and $|x|$ odd. If $|C_G(x)|$ is even then $n_2(C_G(x))$ is even.
- Since $|x|$ is odd, then $n_2(nil_G(x)) = n_2(C_G(x))$, so $|nil_G(x)|$ is even.

Corollary

For any finite group G, $n_G(G)$ is eulerian if and only if $\Gamma(G)$ is eulerian, and $n_G(G)$ is path-eulerian if and only if $\Gamma(G)$ is path-eulerian.

Results of Eulerian Equivalence

This result allows us to state many results about non-nilpotent graphs that have been previously proven [3] about non-commuting graphs. Some of those are the following:

- If $|G|$ is odd, then $n_G(G)$ is eulerian.
- If $|Z(G)|$ is even, then $n_G(G)$ is eulerian.
- $n_G(D_{2m})$ is eulerian if and only if $m = 1$ or m is even.
- The only finite group with a path eulerian non-nilpotent graph is S_3
- The non-nilpotent graphs of non-abelian finite simple groups are never eulerian.

Diameter of Non-nilpotent Graphs

Diameter of a graph

For a graph Γ, the diameter of Γ, denoted $diam(\Gamma)$, is defined as $max(\{d(x,y))|x,y \in V(G)\})$, where $d(x,y)$ is the minimum path length between x and y.

- Complete multipartite graphs clearly have diameter 2.
- Consequently, $diam(n_G(G)) = 2$ for all $\mathfrak{nn}$-groups.
- The authors of [1] conjecture that $diam(n_G(G)) = 2$ for all groups.

Non-nilpotent graphs of diameter 3

Lemma

For $x, y \in G \setminus nil(G)$, $nil_G(x) \cup nil_G(y) = G$ if and only if $d(x, y) > 2$, where $d(x, y)$ is the minimum distance between x and y in $n_G(G)$.

Justification:

- We know $xy \in nil_G(x)$ or $xy \in nil_G(y)$ so either $\langle xy, x\rangle$ or $\langle xy, y\rangle$ is nilpotent.
- $\langle x, y\rangle$ is a subgroup of both of these groups, and so is nilpotent and $d(x, y) > 1$
- If there exists a z that connects x and y, $z \notin nil_G(x)$ and $z \notin nil_G(y)$. This is a contradiction.
- So $d(x, y) > 2$ and the converse holds similarly.

Non-nilpotent graphs of diameter 3

Consider $C_m \rtimes S_4$, where m is odd. In this group $\sigma a \sigma^{-1} = a^{sgn(\sigma)}$, $sgn(\sigma)$ is the sign function for permutations. The sizes of nilpotentizers of elements are as follows:

- $nil(a^i) = \langle a, A_4 \rangle$, order $12m$.
- If $\sigma = (x, y, z)$ for some $x, y, z \in \{1, 2, 3, 4\}$, $nil(a^i \sigma) = \langle a, \sigma \rangle$, order $3m$.
- If $\sigma = (x, y)(z, w)$ for some $x, y, z, w \in \{1, 2, 3, 4\}$, $nil(a^i \sigma) = \langle a \rangle \times \{1, (x, y)(w, z), (x, w)(y, z), (x, z)(w, y)\}$, for all $i \neq 0 \mod m$, order $4m$.
- If σ is an odd permutation $nil(a^i \sigma) = a^k H a^{-k}, 2k \equiv i \mod m, H \simeq D_8$, order 8.
- For all choices of $x, y, w, z \in \{1, 2, 3, 4\}$, $nil((x, y)(w, z))$ is the entire group minus 3-cycles, order $16m$.

Non-nilpotent graphs of diameter 3

Theorem

For any odd m, $n_G(C_m \rtimes S_4)$ has diameter 3.

Justification:

- Clearly, the only element we must consider are a^i and $(x, y)(w, z)$.
- Each of these classes share a common nilpotentizer.
- We can connect any two powers of a through a transposition and any two elements of the form $(x, y)(w, z)$ through a 3-cycle.
- Transpositions and 3-cycles are connected, so this gives us a path of length at most 3 between any two elements.
- Thus the graph has diameter 3.

Future Work

- Determine a bound on the genus of non-nilpotent graphs.
- Explore cases where non-nilpotent graphs of semi-direct products are not complete multi-partite
- If $n_G(G) \simeq n_G(H)$, does $|G| = |H|$?
- What is a sharp upper bound for the diameter of non-nilpotent graphs?

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