# A Geometric and Analytic Examination of a Midpoint Iterative Function System

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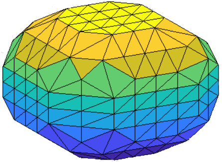


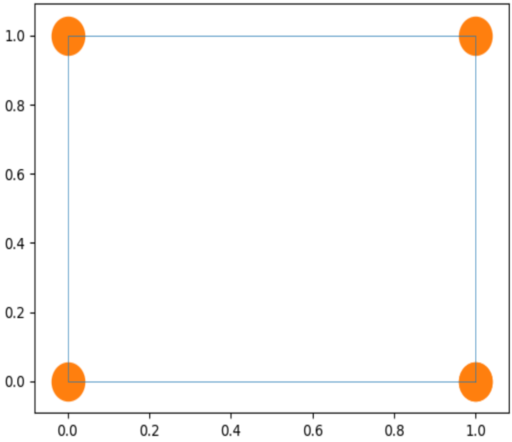
# Outline

# The Midpoint Process

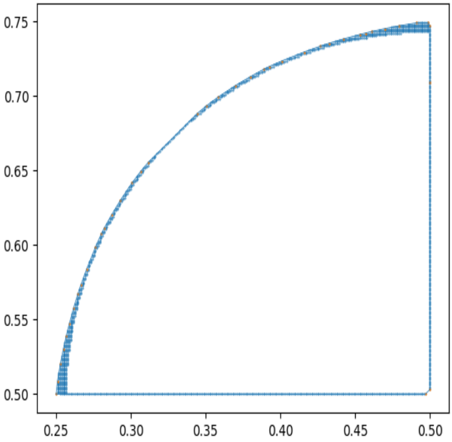
## Simple Definition

1. Consider the set $P_0$ consisting of the vertices of the unit square: (0,0),(0,1),(1,0),(1,1).
2. Generate the midpoint of every pair of points in $P_0$ and consider this set of midpoints to be the first iteration, $P_1$.
3. Repeat from $P_n$ to $P_{n+1}$

# Iterations 0 - 7 (Outer Ring)



# Iteration 10



# Formal Definitions

## $\mathscr{A}$ and the Metric Space

Let $I$ be the unit interval and let $I^2 := I \times I$. Let $\mathscr{A} = \{A \subset I^2 \mid A \text{ is compact}\}$. For $A \in \mathscr{A}$, and $\epsilon > 0$, let $U(A, \epsilon)$ be the $\epsilon$-neighborhood of $A$.

**Definition**

For $A, B \in \mathscr{A}$, define the Hausdorff metric:
$\rho(A, B) = \inf\{\epsilon > 0 \mid A \subset U(B, \epsilon),\ B \subset U(A, \epsilon)\}$

The metric space $(\mathscr{A}, \rho)$ is complete ; see Munkres [Mun]. Note: in $(\mathscr{A}, \rho), \lim_{n\to\infty} Conv(P_n) = \mathcal{P}$

# Formal Definitions

## Defining the IFS $\Phi$

Let $P_0 = \{(0,0),(1,0),(0,1),(1,1)\}$. For $i \in \mathbb{N}$, we iteratively define a collection of sets $\{P_i\}$ such that $v^{(i)} \in P_i$ if and only if $v^{(i)} = \frac{1}{2}(v_1^{(i-1)} + v_2^{(i-1)})$, for some $v_1^{(i-1)}$, $v_2^{(i-1)} \in P_{i-1}$.

### Definition

$\Phi = \{\phi_i\}, i \geq 0$, in which $\phi_i : P_i \times P_i \longrightarrow P_{i+1}$ is given by $\phi_i(v_1^{(i)}, v_2^{(i)}) = \frac{1}{2}(v_1^{(i)} + v_2^{(i)})$.

# Midpoint IFS Iterations 0 - 5

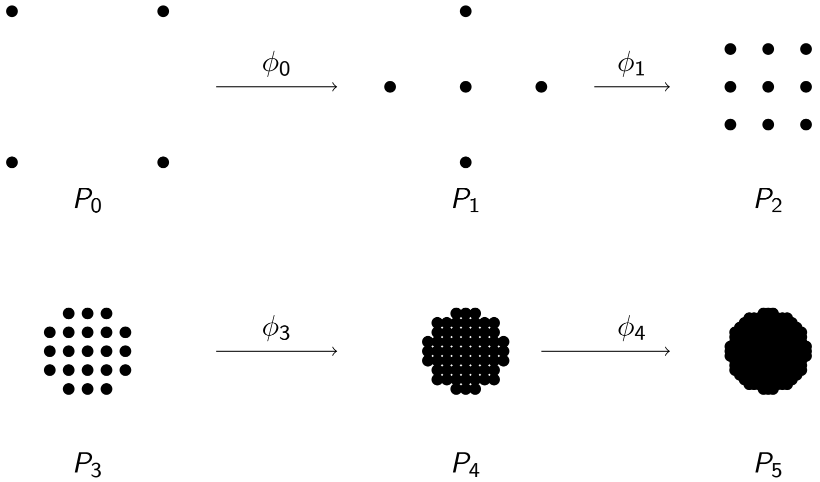



Figure: Illustration of the action of $\phi_i$ $(0 \leq i \leq 4)$ in a consecutive manner.

# Formal Definitions

## Set Notations

**Definition**

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Let $\mathcal{P} = \bigcap_{i=0}^{\infty} \mathrm{Conv}(P_i)$.

**Definition**

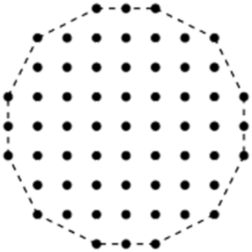
Let $\mathcal{B}$ be the boundary of $\mathcal{P}$.

# Set Notations (Cont.)

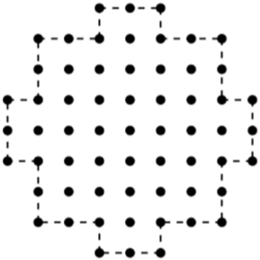
**Definition**

The smallest closed polygon containing a set $P_i$ is defined as the **iteration boundary**.

# Definition Illustrations



Convex Hull
(With Boundary Shown)



Iteration Boundary

# Formal Definitions

## Point Classifications

**Definition**

If $x \in P_i \cap P_{i+1}$, then $x$ is **immortal** $\forall i$.

**Definition (Alternate)**

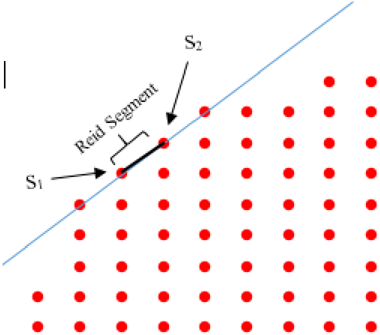
If $x_1, x_2, x_3 \in P_i$ are collinear such that $x_2 = \frac{1}{2}(x_1 + x_3)$, then $x_2$ is **immortal**.


Figure: Illustration of Alternate Definition: $x_2$ is immortal.

# Definition of Reid Segment

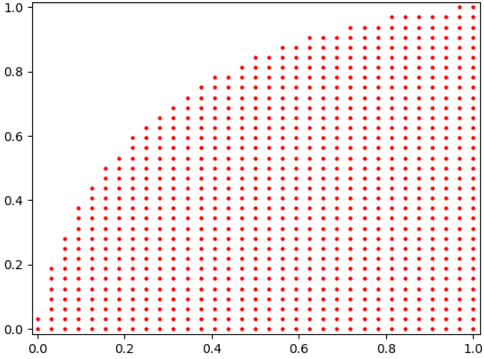
**Definition**

A line containing two Cauchy sequences of mortal points converging to points $s_1^{(i)}$, $s_2^{(i)}$ on $\mathrm{Conv}(P_i)$ defines a **Reid Segment** between $s_1^{(i)}$, $s_2^{(i)}$.



# **Case I** and **Case II**.

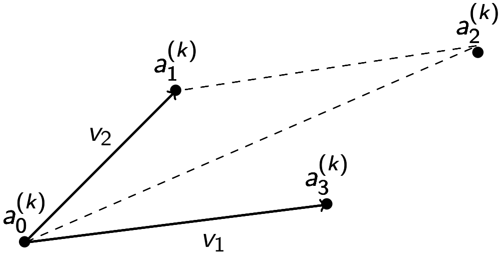
Examination of computer modeling of the development of the IFS $\Phi$ identifies two unique, exhaustive domains which complete the iterative development of $B_i$'s. For ease, consider all $B_i$'s centered on the origin, and partition the figure such that quadrant **II** is the segment of focus. Let it be positioned on $[0,1]^2$ as shown below. When we refer further to $\mathcal{B}$, it is segmented to this domain.



# Case I

## Definition

Consider two positive vectors, $v_1$ and $v_2$. Let $a_0$ be the origin of these two vectors and consider the closure of the set described by their fundamental parallelogram. We assume $a_0$ to be an immortal point, the point representative of $v_1$, $a_1$, to be mortal, and the point representative of $v_1 + v_2$, $a_2$, to be immortal. Further restrict the IFS to the closed domain bounded on the right by the vector $v_1 + v_2$. These conditions define **Case I**.



# Case I

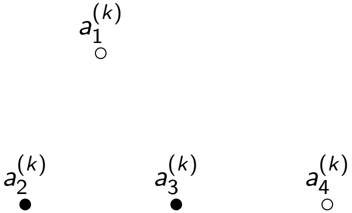
## Definition (Cont.)

It is essential we assume that there exists no point at the bisection of the diagonals of the parallelogram. Since all points in any iteration set $P_i$ must be on a square lattice, this assumption is fulfilled for our analysis.

# Case II

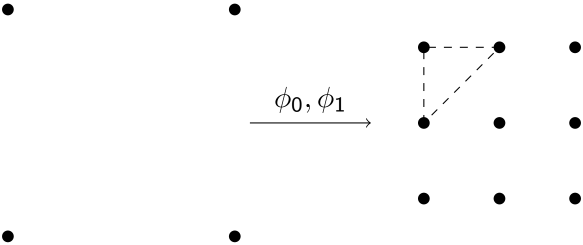
## Definition

**Case II** is defined by the existence of a mortal point, $a_1^{(k)}$, and three points collinear with each other and not with $a_1^{(k)}$. Let the three points be $a_2^{(k)}$, $a_3^{(k)}$, $a_4^{(k)}$ such that $a_2^{(k)}$ is immortal and $a_4^{(k)}$ is mortal.



# Case I

After two iterations, the first **Case I** domain forms.



# Evolution of $\mathcal{B}$

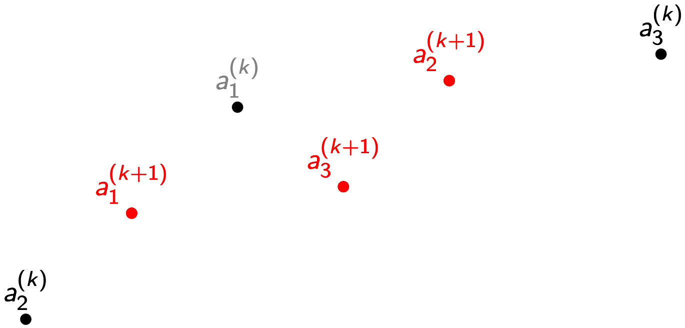
**Proposition**

The existence of **Case I** implies the existence of two instances of **Case II**

**Proposition**

The existence of **Case II** implies the existence of one instance of **Case I**.

# Evolution of Case I



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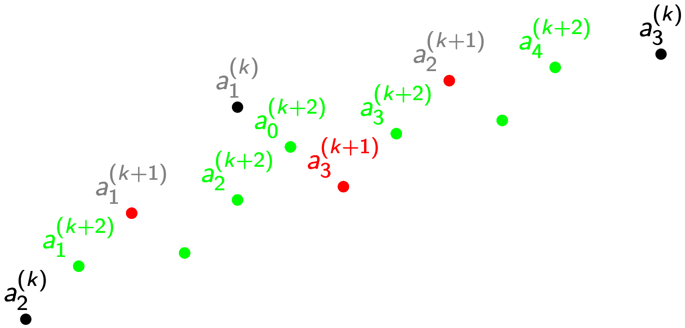
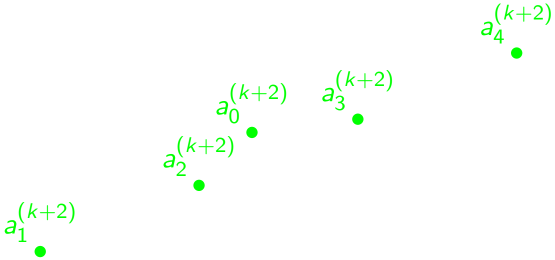


# Evolution of Case I



# Evolution of **Case I**

The figure below demonstrates how **Case I** leads to the emergence of one Reid segment and two **Case II** domains.

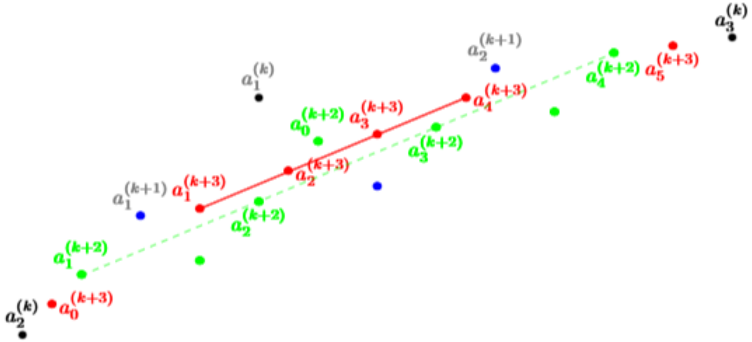



Figure: Emergence of one Reid Segment.

# Evolution of **Case I** (Cont.)

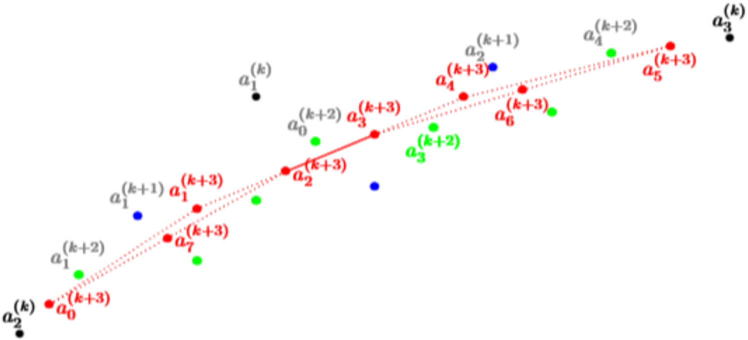
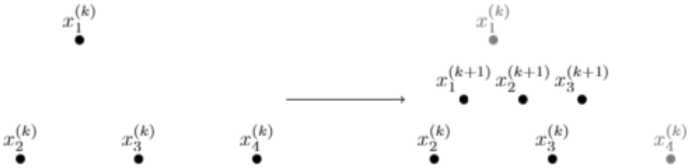



Figure: Emergence of two **Case II** domains.

# Evolution of **Case II**




Select points $a_2^{(k)}$, $a_1^{(k+1)}$, and $a_2^{(k+1)}$. This is acceptable as $a_2^{(k)}$ and $a_1^{(k+1)}$ are immortal. These points satisfy the criteria laid out by the initial assumptions regarding **Case I**. There will arise a Reid Segment in the interior of the domain defined by these three points.

# Hausdorff Measure

The Hausdorff Outer Measure is widely used when classifying fractals, and we adopt the following definition from [Briggs]:

## Hausdorff Outer Measure

Let $S$ be any subset of $X$, and $\delta > 0$ be a real number. We define the Hausdorff Outer Measure of dimension $d$ bounded by $\delta$, $\mathscr{H}_\delta^d$ by:

$$\mathscr{H}_\delta^d(S) = \inf\{\sum_{i=1}^{\infty}(\operatorname{diam}(U_i)^d \mid \bigcup_{i=1}^{\infty} U_i \subseteq S, \operatorname{diam}(U_i) < \delta\}.$$

If we allow $\delta$ to approach 0, then $\lim_{\delta\to 0} \mathscr{H}_\delta^d(S) = \mathscr{H}^d(S)$.

# Hausdorff Dimension of $\mathcal{B}$

Numerical analysis involving the Minkowski-Bouligand box-counting method supported the conclusion that $\dim_H(\mathcal{B}) = 1$. Proving B is diffeomorphic to a circle would prove this, which our paper provides.

# Hausdorff Measure of $\mathcal{P}$

**Proposition**

The set $\mathcal{P}$ has positive Lebesgue measure.

*Proof.* This follows from the existence of four unique, non-collinear, immortal points in $(P_n), n > 3$. There will be formed a line segment $l_i$ between each pair of points such that for any point $x_i \in \mathrm{l}_i$, $x_i$ is immortal and in the limit set. It holds that the limit set has Lebesgue measure greater than zero.

# Disappearing Vertices

We use the following lemma to prove propositions involving bounds on $\mathcal{B}$ and $\mathcal{P}$.

**Lemma**

Any point $x_1$ which is a vertex of $\text{Conv}(P_n)$ will not exist in $P_{n+1}$.

*Proof.* Consider a line tangent to $\text{Conv}(P_n)$ which contains $x_1$. By definition, this line contains no other points in $P_n$. Further, no points exist above this line from $P_n$. Since no two points in $P_n$ are co-linear and contain $x_1$ as a midpoint, $x_1$ will not exist in $P_{n+1}$.

# Bound on $\mathcal{B}$

**Proposition**

The length of $\mathcal{B}$ is bounded above by 1.545.

*Proof.* As n $\Rightarrow \infty$, $(P_n)$ becomes defined through iteration by the previous lemma regarding the disappearance of vertices. By the triangle inequality, the length of $\text{Conv}(P_n + 1) \leq \text{Conv}(P_n)$. The above bound is the length of $(P_{14})$ estimated. Further calculation can lead to further refinement.

# Bound on $\mathcal{B}$

**Proposition**

The set $\mathcal{B}$ is bounded by two concentric circles with radii $1/4$ and $\frac{11\sqrt{2}}{64}$.

# Further Bounds

**Proposition**

The area of $\mathcal{P}$ is bounded above by 0.1885.

*Proof.* As $\Phi$ is a contraction on $P_n$, the previous proposition states $\pi(\frac{11\sqrt{2}}{64})^2 \leq \mathscr{H}^2(P_n) \leq \pi(\frac{1}{4})^2$. Monte Carlo estimation refines this to the above bound, with an error of $\pm 0.0001$.

# Set Properties

We first seek to enumerate as many properties concerning the cardinality and topology of the sets $R$ and $\overline{V}$ as possible.

Let $\mathcal{B}$ have the standard topology, $\mathscr{T}$. Let $R$ be the set of Reid segments, each open with respect to $\mathscr{T}$, and $V$ to be the set of all vertex points that arise from **Case II** domains and endpoints of Reid segments that arise from **Case I** domains. We must consider $\overline{V}$, i.e. the closure of $V$ at the limit of the iterative system.

# Cardinality and Topology

**Theorem**

The set $R$ is countable.

*Proof.* Without loss of generality we demonstrate that the cardinality of Reid segments in the upper-left quadrant is countable. Let $r \in R$ be a Reid segment in this region and let $r = \mathrm{int}\{(1-\lambda)x_1 + \lambda x_2 \mid \lambda \in [0,1]\}$. Consider the projection map $f : I^2 \longrightarrow I$ from $R$ to its set of x-coordinates. The image of $f$ is a collection of disjoint open sets, which is countable, as $I$ is at most a countable collection of disjoint open sets.

# Cardinality and Topology

**Theorem**

The set $\overline{V}$ is complete.

*Proof.* We use the boundary's self-similarity. It suffices to show that the four Cauchy sequences uniquely bounded by any **Case I** domain and whose limit points are elements of $\overline{V}$. The completeness of $\overline{V}$ follows from the convergence of these Cauchy sequences.

# Cardinality and Topology

**Theorem**

The set $\overline{V}$ has no isolated points and is uncountable with respect to the boundary.

# Cardinality and Topology

**Theorem**

The set $\overline{V}$ is nowhere dense.

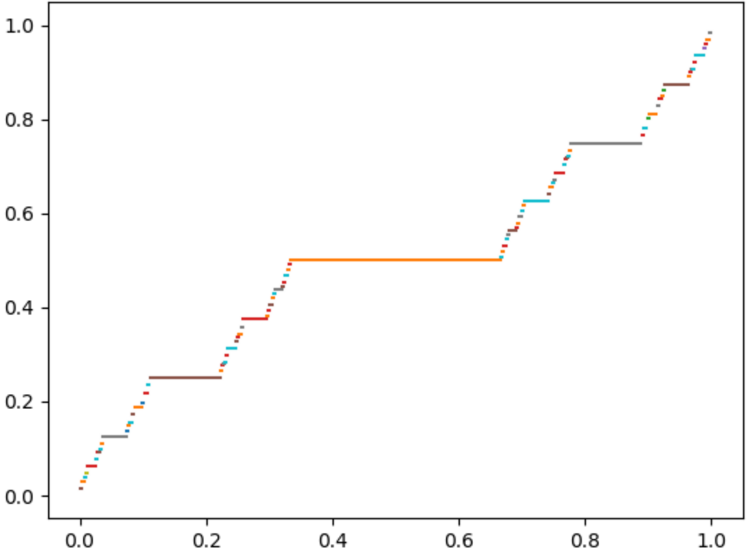
*Proof.* $(\overline{V})^c = R$, therefore $\mathrm{int}(\overline{V}) = \emptyset$.

# Cardinality and Topology

**Theorem**

The set $\overline{V}$ is homeomorphic to the Cantor ternary set.

# Fractal Structure



# Fractal Structure (Cont.)

**Falconer**'s Definition of a Fractal

- F has a fine structure, i.e. detail on arbitrarily small scales.
- F is too irregular to be described in traditional geometrical language, both locally and globally.
- Often F has some form of self-similarity, perhaps approximate or statistical.
- Usually, the 'fractal dimension' of F (defined in some way) is greater than its topological dimension.
- In most cases of interest F is defined in a very simple way, perhaps recursively.

Note: (4) is **Mandelbrot**'s definition of a fractal.

# Farey Sequences

**Definition**

A **Farey sequence** $F_n$ of order $n$ gives all rational numbers $\frac{p_n}{q_n} \in [0, 1]$ such that $q_n \leq n$.

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It follows from the development of **Case I**'s and **Case II**'s that all rational tangent slopes exist on $\mathcal{B}$.
The monotonicity and rationality of the convex boundary of our system implies that the set $S_n$ of tanget slopes of $\mathcal{B}$ is a proper subset of $F_n$ for any iteration $n$.

# Stern-Brocot Tree


Figure: Stern-Brocot Tree

# Continuity of the Derivative

**Proposition**

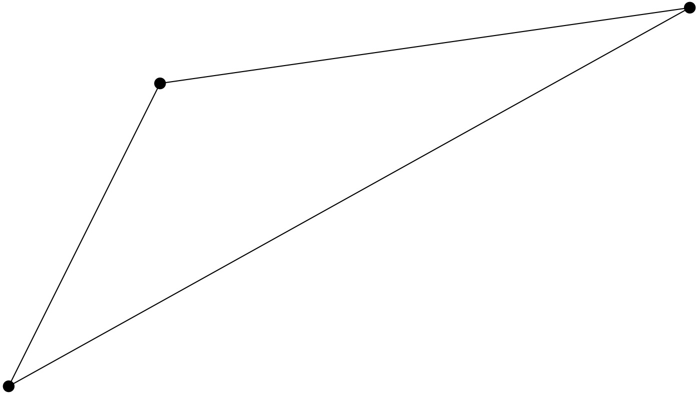
$\mathcal{B}$ is a $\mathcal{C}^1$ function whose derivative is a 'devil's staircase-like' function.

We show in our paper that the derivative of $\mathcal{B}$ is:

- Continuous
- Exists and is positive everywhere
- Non-increasing and non-decreasing everywhere

# Continuity of the Derivative

## Sketch of Proof



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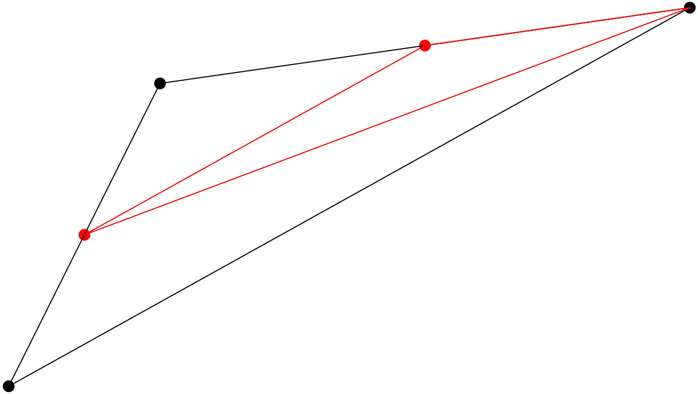

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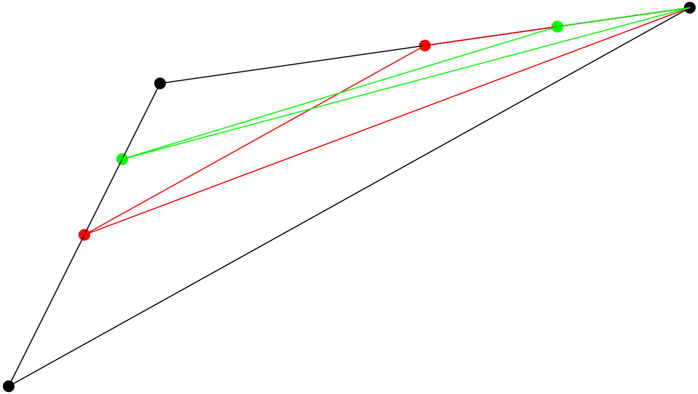

# Moving Forward

- Arithmetic definition of $\mathcal{B}$
- Other dimensions
- $P_0$s of other shapes
- Hausdorff dimension of $\mathcal{B} \cup \left(\bigcup_{i=1}^{\infty} B_i\right)$

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