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November 2024

Let's take a deeper look at traffic!

What do our cities do when we complain about traffic?

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Build a new road!!

But sometimes...

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We get this anyways:

Deep Dive: Traffic

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So, what in God's green earth is going on here?

The answer:

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The answer: Math!

Braess's paradox is a counter-intuitive theory in traffic flow in which the addition of roadways to an already existing network can introduce unstable equilibrium- BLA BLA BLA

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Essentially, adding roads = bad news bears (sometimes)

But what does this have to do with math?

Why understand Braess's Paradox?

- Traffic Engineering and Urban Planning
  - Road Network Design
  - Traffic Simulation Models
- Telecommunication Networks
  - Internet and Data Routing
- Supply Chain Management
  - Healthcare, Power grids, Warehouses, etc.

Anything you can imagine that includes transportation, Braess has his grimy little fingers all over.

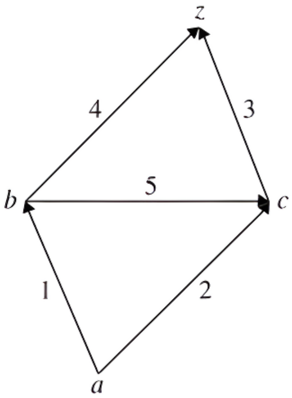
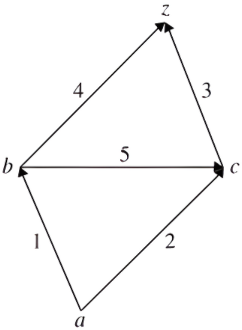


Figure: This is a doohickey

Some guidelines about this toy model are as follows:

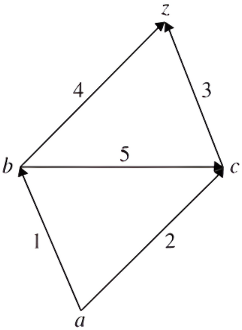
- 1.) Drivers start from node a and can only drive in the direction of the arrows
- 2.) It is assumed each driver is aware of the route times
- 3.) Each driver picks independently of other drivers
- 4.) The time to travel for each road is denoted by $t_i(\phi)$, where i is the specific road and ϕ is the number of drivers on said road
- 5.) The total number of drivers on the model is denoted by $|\Phi|$



Each side is assigned a time function. For simplicity's sake, they are assigned a linear function:

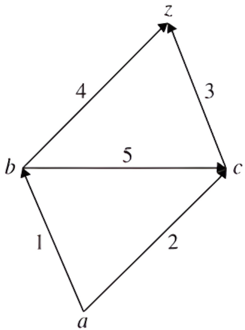
- Side 1: $t_1(\phi) = 10\phi$
- Side 2: $t_2(\phi) = 50 + \phi$
- Side 3: $t_3(\phi) = 10\phi$
- Side 4: $t_4(\phi) = 50 + \phi$
- Side 5: $t_5(\phi) = 10 + \phi$

(Notice that side 5 is a "shortcut")



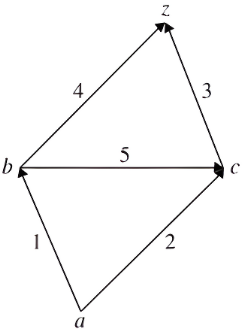
To do some math with this, let's declare a couple more variables:

- x denotes the number of drivers who use path *abz*
- y denotes the number of drivers who use path *acz*
- z denotes the number of drivers who use path *abcz*



Then, we have the following functions to describe travel times:

- $T_{abz} = 10(x+z) + (50+x)$
- $T_{acz} = 10(y+z) + (50+y)$
- $T_{abcz} = 10(x+z) + (10+z) + 10(y+z)$



Let's not forget what our end goal is here:

To optimize travel times with and without a shortcut road for all drivers given our previous rule set that people are selfish and compare them to eachother.

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So how can we optimize such gross, multi-variable functions?

Everyone's favorite: Lagrange multipliers!

- Lagrange Multipliers are a way to find extrema given an objective and constraint function
- We construct the Lagrangian function, denoted by $\mathcal{L} = f(x_1, x_2, ..., x_n) - \lambda(g(x_1, x_2, ..., x_n))$
  - λ is our scaling factor, i.e., our Lagrange Multiplier
  - $f(x_1, x_2, ..., x_n)$ is our objective function
  - $g(x_1, x_2, ..., x_n)$ is our constraint function

- To find the extrema, we need to take partial derivatives with respect to each variable and the Lagrange Multiplier
  - This is denoted by:
$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, \ \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$
  - Doing so yields the system:
    - $\nabla f(x_1, x_2, ..., x_n) = \nabla \lambda g(x_1, x_2, ..., x_n)$
    - $g(x_1, x_2, ..., x_n) = 0$

So, we first need everything you would need for a normal Lagrange Multipliers problem:

- An objective function
- A constraint function

Let's start by noting that our constraint function is going to be limited by the number of drivers, i.e., it can be given by:

$$x + y + z = |\Phi|$$

Now that we have $x + y + z = |\Phi|$, lets construct our objective function *with* the shortcut. It will be given by:

$$A(x, y, z) = \frac{x \cdot T_{abz} + y \cdot T_{acz} + z \cdot T_{abcz}}{|\Phi|}$$

Applying Lagrange Multipliers

Let's apply the method we laid out before:

$$\begin{cases} A_x = \frac{22x}{|\Phi|} + \frac{20z}{|\Phi|} + \frac{50}{|\Phi|} = \lambda \\ A_y = \frac{22y}{|\Phi|} + \frac{20z}{|\Phi|} + \frac{50}{|\Phi|} = \lambda \\ A_z = \frac{20x}{|\Phi|} + \frac{20y}{|\Phi|} + \frac{42}{|\Phi|} + \frac{10}{|\Phi|} = \lambda \\ g(x,y,z) = x + y + z = |\Phi| \end{cases}$$

To solve this, we can slap it in a calculator because no one wants to do Linear Algebra by hand. That is,

We Have Technology!

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Our matrix,

$$\begin{bmatrix} \frac{22}{|\Phi|} & 0 & \frac{20}{|\Phi|} & -1 & -\frac{50}{|\Phi|} \\ 0 & \frac{22}{|\Phi|} & \frac{20}{|\Phi|} & -1 & -\frac{50}{|\Phi|} \\ \frac{20}{|\Phi|} & \frac{20}{|\Phi|} & \frac{42}{|\Phi|} & -1 & -\frac{10}{|\Phi|} \\ 1 & 1 & 1 & 0 & |\Phi| \end{bmatrix},$$

has an *RREF* of:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{11|\Phi|-20}{13} \\ 0 & 1 & 0 & 0 & \frac{11|\Phi|-20}{13} \\ 0 & 0 & 1 & 0 & \frac{40-9|\Phi|}{13} \\ 0 & 0 & 0 & 1 & \frac{2(31|\Phi|+505)}{13|\Phi|} \end{bmatrix}.$$

Applying Lagrange Multipliers

The *RREF* implies that:

$$x = \frac{11|\Phi| - 20}{13},\ y = \frac{11|\Phi| - 20}{13},\ \text{and } z = \frac{40 - 9|\Phi|}{13}.$$

Plugging these values into $A(x, y, z)$ yields

$$\frac{31|\Phi|^2 + 1010|\Phi| - 800}{13|\Phi|},$$

which is our optimized objective function.

Applying Lagrange Multipliers

Now that we have optimized the objective function with roads, we need to replicate the same process without roads.

Let's first note that in a model without the shortcut, there would be 0 z drivers, which we can write as:

$$A(x, y, 0) = \frac{11x^2+50x+50y+11y^2}{|\Phi|}.$$

Repeating the same optimization process as before yields:

$$x = y = \frac{|\Phi|}{2}.$$

Plugging this into $A(x, y, 0)$ yields:

$$\frac{11|\Phi|+100}{2}.$$

Now, let's directly compare our two results of

$$A(x,y,z) = \frac{31|\Phi|^2+1010|\Phi|-800}{13|\Phi|} \text{ and } A(x,y,0) = \frac{11|\Phi|+100}{2}.$$

First, we see the the two functions approach equality for the values $0 < |\Phi| < 4.\overline{4}$, so let's explore that further.

Setting the equations equal to each other yields that $|\Phi| = 4.\overline{4}$. This appears since at $|\Phi| = 4.\overline{4}$ our z value $z = \frac{40-9|\Phi|}{13}$ in our shortcut becomes zero, and thus equal to our path without a shortcut, $A(x, y, 0)$.

Moreover, for values above $4.\overline{4}$, we see our z path becomes negative, which makes no sense.

So, what does this mean for us?

First, let us note that we can't send less than half of a person down any given route, so we will round this value down to 4. Then, for $|\Phi| \geq 4$, we need to find different equations to compare, but what?

Let us also notice that the function for our path $T_{abcz}(x, y, 0)$, the time of the shortcut when zero people are on it, is given by $10x + 10y + 10$, under the constraint $x + y = |\Phi|$. Then, the same path is also given by $10|\Phi| + 10$ when using $y = |\Phi| - x$.

Applying this, we see that for values of $|\Phi| < 9$, our shortcut path with no drivers yields an average minimum time always less than $\frac{11|\Phi|+100}{2}$, the average minimum time for $A(x, y, 0)$. Moreover, $|\Phi| \geq 9$ will always yield a slower average minimum time.

Interpreting this tells us that, on our toy model, 9 drivers and beyond will cause the paradox to disappear, and the same goes for $|\Phi| < 4$.

So, what have we learned?

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In essence, when concretely exploring Braess's paradox, we can conclude that there will be a specific number of drivers that dictates the appearance and disappearance of Braess's paradox (given the stipulation that drivers are aware of the route times). In our specific example, we see the paradox is not present for $|\Phi| < 4$ and $|\Phi| \geq 9$, while it is present for $4 \leq |\Phi| < 9$.

History Lesson (Gross)

Though a group of researchers concluded that Braess's paradox is just about as likely to occur as not to occur when a new road is introduced, it has been observed many times in the real world, so let's see a couple of examples!

Cheonggyecheon Restoration

- Seoul demolished the Cheonggyecheon elevated highway and restored the underlying river as part of an urban revitalization project. Many anticipated that removing the highway would worsen traffic congestion. Surprisingly, traffic improved after the demolition, as it encouraged more efficient use of existing roadways, public transportation, and alternative routes.

1990 Earth Day closure

- The 1990 Earth Day closure of 42nd Street in Manhattan demonstrated Braess's paradox, showing that limiting road access can sometimes improve overall traffic flow. With the street closed, drivers had to use alternative routes, which unexpectedly reduced congestion in the area.

Sources

- Knödel, W. (31 January 1969). Graphentheoretische Methoden Und Ihre Anwendungen. Springer-Verlag. pp. 57–59. ISBN 978-3-540-04668-4.
- Easley, D.; Kleinberg, J. (2008). Networks. Cornell Store Press. p. 71.
- Braess, D. (December 1968). "Über ein Paradoxon aus der Verkehrsplanung". Unternehmensforschung Operations Research - Recherche Opérationnelle. 12 (1): 258–268. doi:10.1007/bf01918335. ISSN 0340-9422. S2CID 39202189
- https://www.nytimes.com/1990/12/25/health/what-if-they-closed-42d-street-and-nobody-noticed.html