CONVERGENCE OF A GENERALIZED MIDPOINT ITERATION

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Abstract. We give an analytic proof for the Hausdorff convergence of the midpoint or derived polygon iteration. We generalize this iteration scheme and prove that the generalization converges to a region of positive area and becomes dense in that region. We speculate on the centroid or derived polyhedron iteration.

1. Background

First we justify, as an exercise, a few standard results about nested compact sequences. Then, we examine the midpoint iteration scheme for convex polygons, with remarks about concave starting conditions and regularity in the limit.

The convergence behavior of the midpoint iteration has been extensively studied [4]. Our ultimate goal is to define a generalization of the midpoint procedure on the plane and prove similar convergence results. This new iteration will be characterized by an increasing number of vertices at each step. Our main result is the convergence of these finite sets of vertices to a dense set of positive area.

Finally, we speculate on the centroid iteration for polyhedra and prove that the limit is a set of positive volume.

2. Definitions and Conventions

Let $d(\cdot,\cdot)$ denote the Euclidean distance between two points in $\mathbb{R}^2$ and $|\cdot|$ the Euclidean norm. Let $c(\cdot)$ denote the convex hull of a set. Denote set closure, with respect to the standard metric topology by $cl(\cdot)$ and the open ball of radius $\varepsilon > 0$ about x by $B(x, \varepsilon)$.

We identify a polygon with a convex hull of a finite number of affine independent points on the real plane. Such a convex hull is necessarily bounded and closed. A finite set of points, or *vertices*, are in *general linear position* if no three distinct elements of the set are collinear.

3. Compactness of Polygons

Our iteration procedures will deal with sequences of subsets decreasing under set inclusion. There are many standard results about these *nested* sequences of compact subsets which can be applied to polygons on the plane. We prove some here as an exercise for ourselves.

Suppose $\{K_n\}_{n\in\mathbb{N}}$ is a nested sequence of compact subsets of the plane, such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$. By the Heine-Borel characterization of compact sets in $\mathbb{R}^2$, each set is equivalently closed and bounded.

Proposition 3.1. *Suppose $\{K_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty compact sets. Then the intersection*

$$I = \bigcap_{n\in\mathbb{N}} K_n$$

is nonempty and compact.

Proof. Define a sequence $s = \{s_n\}_{n\in\mathbb{N}}$ such that $s_i \in K_j \setminus K_{j-1}$ if and only if $i = j$. Each compact $K_j \setminus K_{j-1}$ contains finitely many terms s_n of the sequence.

We can identify a convergent subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ of s with limit $\sigma \in K_1$ (and so in all K_n), by compactness. If there exists an N such that $m > N$ implies $\sigma \notin K_{m+1} \setminus K_m$, then σ would be a limit point in $K_N \setminus K_{N-1}$, contradicting the fact that only finitely many s_{n_k} are in $K_N \setminus K_{N-1}$. So I is nonempty.

In addition, I is bounded and closed, since $I \subset K_1$ and arbitrary intersections of closed sets are closed. Hence, I is compact in $\mathbb{R}^2$. □

Besides the usual area and perimeter, the diameter is another useful characteristic of a polygon which we will use to prove convergence results.

Definition 3.2. Let diam(K) be the diameter of a set K. That is,

$$\mathrm{diam}(K) = \sup_{x,y\in K} d(x, y).$$

Proposition 3.3. *If K is compact in $\mathbb{R}^2$, then diam(K) is finite.*

Proof. K is bounded, and there exists a point $p \in \mathbb{R}^2$ and $M > 0$ such that $d(x, p) \le M$ for all $x \in K$. Then, by the triangle inequality,

$$\mathrm{diam}(K) \le 2M.$$

□

Note that the previous proposition depends only on the boundedness of a compact set K.

There is an immediate connection between the sequence of diameters of nested compact sets and the diameter of their intersection.

Proposition 3.4. *Let $r \ge 0$. If diam$(K_n) \ge r$ for $n \in \mathbb{N}$, then diam$(I) \ge r$.*

Proof. (Sketch) We claim that this proposition follows from the discussion in Propositions 3.8 and 3.9 on convergent subsequences of nested compact sets. For brevity, we omit a rigorous proof.

□

Proposition 3.5. *Let $I = \bigcap_{n\in\mathbb{N}} K_n$ be the intersection of K_n. If $\lim$ diam$(K_n) = 0$ then $I = \{x_0\}$ for some $x_0 \in K_1$.*

Proof. Suppose that I is not a singleton. So diam$(I) \ne 0$, and $I \subset K_n$ for all n implies $\lim \mathrm{diam}(K_n) > 0$. □

Corollary 3.6. *The diameters of the K_n converge to the diameter of their intersection; that is,*

$$\lim \mathit{diam}(K_n) = \mathit{diam}(I).$$

Proof. In Proposition 3.4, let $r = \lim \operatorname{diam}(K_n)$, since $\operatorname{diam}(K_n)$ is a non-increasing sequence. So $\lim \operatorname{diam}(K_n) \leq \operatorname{diam}(I)$. But $\operatorname{diam}(K_n)$ bounds $\operatorname{diam}(I)$ for arbitrary n since $I \subset K_n$; so there is equality. □

We can also justify in interchanging the limits in these propositions because $\operatorname{diam}(\cdot)$ can be proven to be a continuous function with respect to the Hausdorff metric, which we will now introduce.

We identify a convex polygon with the convex hull of a finite set $V \subset \mathbb{R}^2$ of vertices in general linear position and seek an appropriate sense of convergence. The Hausdorff metric allows us to define convergence such that if K_n limits to K, the points of K_n become arbitrarily close to their nearest neighbors in K.

Definition 3.7. The Hausdorff distance of two nonempty compact subsets A and B of a metric space is defined to be

$$d_H(A, B) = \sup \left\{ \sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(b, a) \right\}.$$

A sequence of nonempty compact subsets $\{K_n\}$ *converges in the Hausdorff metric* to K if $\lim d_H(K_n, K) = 0$.

The geometric iterations in the following sections produce sequences of nested compact subsets of $\mathbb{R}^2$. We will prove convergence results with respect to this Hausdorff distance; therefore, whenever we discuss the limit a of sequence of sets, we mean that the sequence comes within every ε-ball of the limit set with respect to the Hausdorff distance. (That is, limits of sequences of sets are taken with respect to the metric topology induced by the Hausdorff distance on the set of compact subsets of $\mathbb{R}^2$.)

Proposition 3.8. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of compact subsets such that $K_{n+1} \subset K_n$ for all $n \in \mathbb{N}$, and I their infinite intersection. Then*

$$\lim d_H(K_n, I) = 0.$$

Proof. Since $I \subset K_n$ for all $n \in \mathbb{N}$ and so $\sup_{y \in I} \inf_{x \in K_n} d(y, x) = 0$, we focus on the quantity

$$\sup_{x \in K_n} \inf_{y \in I} d(x, y) = d_H(K_n, I).$$

By compactness, there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

1. $x_n \in K_n \setminus I$
2. $y_n \in I$
3. $d(x_n, y_n) = \sup_{x \in K_n} \inf_{y \in I} d(x, y)$

We can identify two convergent subsequences $\{x_{n_k}\}_{k \in \mathbb{N}}$ and $\{y_{n_k}\}_{k \in \mathbb{N}}$, with respective limits α and β, such that for all k,

$$d(x_{n_k}, y_{n_k}) = \sup_{x \in K_{n_k}} \inf_{y \in I} d(x, y).$$

by a subsequence index argument. Now, $\alpha \in I$ necessarily. To see this, suppose for contradiction that $\alpha \in K_n \setminus I$ for some n. Then α is a point of accumulation and there exists an $\varepsilon > 0$ such that the open ball $B(\alpha, \varepsilon) \subset K_n \setminus I$ contains infinitely many points of $\{x_n\}$. Then $K_n \setminus I$ contains some x_m for $m \neq n$, contradicting our hypotheses.

Therefore

$$d(x_{n_k}, y_{n_k}) = \inf_{y \in I} d(x_{n_k}, y) \leq d(x_{n_k}, \alpha).$$

Passing to the limit, we have $\lim_k d(x_{n_k}, y_{n_k}) = 0$. □

A similar property holds for countable unions of compact sets:

Proposition 3.9. *Suppose $\{A_n\}$ is a sequence of compact sets such that $A_n \subset A_{n+1}$ and $A_n \subset K$ for all $n \in \mathbb{N}$ and some compact K. Let $U = \cup_n A_n$. Then $\lim A_n = cl(U)$.*

Proof. Since A_n is a subset of K, $cl(U)$ is a bounded and closed set. So identify two subsequences $\{a_k\}$ and $\{b_k\}$ in the compact $cl(U)$ such that

1. $a_k \in U$
2. $b_n \in A_n \setminus A_{n-1}$
3. $d(a_n, b_n) = \sup_{x \in \mathcal{U}} \inf_{y \in A_n} d(x, y)$

Then the same style of proof as in the previous proposition with the following inequality yields the result

$$d(b_k, a_k) = \inf_{b \in A_k} d(b, a_k) \leq d(\alpha, a_k).$$

□

4. Midpoint Iteration

First solved by the French mathematician J.G. Darboux, the midpoint polygon problem (sometimes called the derived polygon iteration) has been examined using diverse techniques, including finite Fourier analysis and matrix products [1] [2] [3] [4]. We present an elementary analysis proof of the midpoint polygon problem.

For notation, let $P^{(n)}$ be the set of vertices of the nth convex polygon $P^{(n)}$ in the midpoint iteration. Denote the elements of $P^{(n)}$ by $v_i^{(n)}$. Use the iteration scheme

$$v_i^{(n+1)} = \frac{1}{2}(v_i^{(n)} + v_{i+1}^{(n)}).$$

Theorem 4.1. *(Midpoint iteration): Given an initial set of vertices*

$$P^{(0)} = \left\{ v_1^{(0)}, \ldots, v_Q^{(0)} \right\}$$

in general linear position, the sequence of convex hulls $c(P^{(n)})$ produced by midpoint iteration converge in the Hausdorff metric to the centroid of $c(P^{(0)})$.

Proof. Without loss of generality, suppose the centroid of $P^{(0)}$ is the origin. By Proposition 3.8, the Hausdorff limit $\lim P^{(n)}$ is equal to the intersection $\bigcap c(P^{(n)})$, which we know is nonempty and compact.

Consider the sequence of diameters $D = \left\{ \text{diam}(c(P^{(n)})) \right\}_{n \in \mathbb{N}}$. Each diameter

$$\text{diam}(c(P^{(n)})) = \max_{i,j} |v_i^{(n)} - v_j^{(n)}|$$

is the arclength of the maximum line segment contained in $c(P^{(n)})$.

We show by induction that the maximum edge length of $c(P^{(n)})$ tends to zero as n goes to infinity. Examine the edge lengths of $c(P^{(1)})$, where indices are taken modulo Q:

$$|v_i^{(1)} - v_{i+1}^{(1)}| = \frac{1}{2}|v_i^{(0)} - v_{i+2}^{(0)}| \leq \frac{1}{2}\text{diam}(c(P^{(0)}))$$
$$\sup_i |v_i^{(1)} - v_{i+1}^{(1)}| \leq \frac{1}{2}\text{diam}(c(P^{(0)})).$$

Generally, $\sup_i |v_i^{(n)} - v_{i+1}^{(n)}| \leq (\frac{1}{2})^{n-1}\text{diam}(c(P^{(0)}))$. And the perimeter bounds the diameter of a closed convex polygon, so

$$\text{diam}(c(P^{(k)})) \leq \sum_{i=1}^{Q} |v_i^{(n)} - v_{i+1}^{(n)}| \leq Q(\frac{1}{2})^{n-1}\text{diam}(c(P^{(0)})).$$

We conclude $\lim \text{diam}(c(P^{(n)})) = 0$. By Proposition 3.5, $\lim c(P^{(n)}) = \{x_0\}$ for some x_0. But by calculation, we see that the centroid of $P^{(n+1)}$ is identically the centroid of $P^{(n)}$, the origin. So $0 \in \bigcap c(P^{(n)})$ is the limit of $\left\{c(P^{(n)})\right\}$. $\square$

Now we consider the regularity of the iterate polygons. On this topic, Darboux proves:

> *... ils tendent à devenir semblables à des polygones semi-régulièrs inscrits dans une ellipse.* [1]
> ... [the iterates] tend to become semi-regular polygons, each of which is inscribed in an ellipse.

We then reasonably expect the difference between arbitrary edge lengths of some iterate $P^{(n)}$ to tend to zero as n tends to infinity.

Proposition 4.2. *Let $e_j^{(n)}$ denote the jth edge length $|v_j^{(n)} - v_{j+1}^{(n)}|$ of the nth iterate of an initial Q-polygon. Then*

$$\lim_n |e_i^{(n)} - e_j^{(n)}| = 0 \textit{ for any } i, j \textit{ indices modulo } Q.$$

Proof. We see that:

$$|e_i^{(n)} - e_j^{(n)}| \leq 2\text{diam}(P^{(n)})$$

and by the proof of Proposition 3.1, $\lim \text{diam}(c(P^{(n)})) = 0$. $\square$

By the previous proposition, given $\varepsilon > 0$, there is a large enough N such that $n > N$ implies

$$|e_i^{(n)} - e_j^{(n)}| < \varepsilon$$

for all i, j indices taken modulo Q.

5. Generalized Midpoint Iteration

In the previous section, we considered a midpoint iteration combining adjacent vertices of a polygon of the plane. This process yields a natural generalization to a midpoint iteration on finite planar sets which we call the Generalized Midpoint Iteration (GMI).

Informally, the GMI on $P^{(0)}$ produces vertex sets $P^{(n)}$ containing every possible midpoint combination of the previous vertex set $P^{(n-1)}$, and records how many times $p \in P^{(n)}$ occurs as a midpoint by the multiplicity function $\mu_n : P^{(n)} \to \mathbb{N}$.

The GMI is interesting to study because the number of vertices in each iterate grows over time. Experimentally, this causes the GMI to converge to a region of positive area as well as to become dense in this region — more complicated behavior than the midpoint iteration, which converges to a single point.

Now, we formalize this new procedure.

Definition 5.1. (Generalized midpoint iteration): Let

$$P^{(0)} = \left\{p_1^{(0)}, \dots, p_Q^{(0)}\right\} \subset \mathbb{R}^2$$

be an initial set of Q distinct vertices of a convex polygon. Identify $P^{(0)}$ with the vector $V^{(0)} \in (\mathbb{R}^2)^Q$ with entries

$$V^{(0)} = (p_1^{(0)}, \dots, p_Q^{(0)})$$

Inductively define $V^{(n)}$ and $P^{(n)}$ as follows. Let T be any linear map such that

$$T : V^{(n-1)} \mapsto \left\{ \frac{p_j^{(n-1)} + p_i^{(n-1)}}{2} : i \neq j \right\} := V^{(n)}$$

Define $P^{(n)}$ as the set of entries of $V^{(n)}$. Define the *multiplicity* $\mu_n(p)$ of $p \in P^{(n)}$ as the maximal number of times p occurs in the vector $V^{(n)}$ as an entry. Then the indexed collection of tuples:

$$\mathbb{P}^{(k)} = (P^{(n)}, \mu_n)$$

is the *generalized midpoint iteration* of the set $P^{(0)}$.

Now, for a recursive formula of the multiplicity function μ_k, consider any $p \in P^{(k)}$ for some k. Associate to p the set $M(p) \subset P^{(k-1)} \times P^{(k-1)}$ such that

$$M(p) = \left\{ (q_1^{(i)}, q_2^{(i)}) : q_1^{(i)} \neq q_2^{(i)} \text{ and } \frac{q_1^{(i)} + q_2^{(i)}}{2} = v \right\}_{i \in J} \quad (J = \{1, 2, \dots, j_v\})$$

where j_v is the number of such pairs. Next, define a function $\Theta : P^{(k)} \to \mathbb{N}$ by

$$\Theta(p) = \begin{cases} \binom{\mu_{k-1}(p)}{2} & \text{if } p \in P^{(k-1)} \text{ and } \mu_{k-1}(p) > 1 \\ 0 & \text{else} \end{cases}$$

Then, we see that

$$\frac{1}{2} \sum_{i \in J} \mu_{k-1}(q_1^{(i)}) \mu_{k-1}(q_2^{(i)})$$

is exactly the number of times p occurs as an entry $V^{(k)}$ as the the midpoint of distinct points in $V^{(k-1)}$. Similarly, $\Theta(p)$ is the number of times p occurs as an entry as the midpoint of itself in $V^{(k-1)}$. This proves the following proposition:

Proposition 5.2. *Suppose $M(p)$ and $\Theta(p)$ are defined as above for $p \in P^{(k)}$. Then*

$$\mu_k(p) = \frac{1}{2} \sum_{i \in J} \mu_{k-1}(q_1^{(i)}) \mu_{k-1}(q_2^{(i)}) + \Theta(p).$$

The multiplicity $\mu_n(p)$ of a point p produced by the GMI on the nth iteration determines the "longevity" of p. Observe that if $\mu_n(p) > 1$, then p will be an element of $V^{(n+1)}$, and if $\mu_n(p) > 2$, then p will be an element of $V^{(m)}$ forever, for all $m \geq n$.

Proposition 5.3. *If there exists $n \in \mathbb{N}$ such that $\mu_n(p) > 2$ for some $p \in P^{(n)}$, then $p \in P^{(m)}$ for all $m \geq n$.*

Proof. $\mu_n(p) > 2$ implies $p \in P^{(n+1)}$. Then, by Proposition 5.2,

$$\mu_{n+1}(p) \geq \binom{\mu_n(p)}{2} \geq \mu_n(p).$$

This implies $p \in P^{(n+2)}$, since $p \in P^{(n+2)}$ if $\mu_{n+1}(p) > 1$. Passing to the inductive step, $\mu_k(p) \geq \mu_{k-1}(p)$ for all $k > n$; so $p \in P^{(n)}$ for all $k > n$. □

Definition 5.4. For any $n \in \mathbb{N}$, if $p \in P^{(n)}$ satisfies the conditions of Proposition 5.3, then p is a *fixed point* of the GMI on $P^{(0)}$. Denote the set of fixed points of $P^{(n)}$ on the nth step of GMI by $F^{(n)}$.

Recall that in the classical midpoint iteration, the intersection of convex hulls provided the limit set of the polygons. The fixed points described in the previous proposition will play an analogous role in defining the limiting set of convex hulls of $P^{(n)}$ in the generalized midpoint iteration.

Proposition 5.5. *Suppose $p \in P^{(n)}$ is given by $\frac{x+f}{2}$ for some $x \in P^{(n-1)}$ and $f \in F^{(n-1)}$. Then $p \in F^{(n)}$.*

Proof. Calculating the multiplicity of p,

$$\mu_n(p) \geq \mu_{n-1}(x)\mu_{n-1}(f) > 2$$

since $\mu_{n-1}(x) \geq 1$ and $\mu_{n-1}(f) > 2$ □

In the GMI, fixed points always arise quickly if the initial set $P^{(0)}$ contains at least four distinct points.

Proposition 5.6. *Suppose $|P^{(0)}| = 4$, where $|\cdot|$ denotes cardinality of a set. Then $|F^{(n)}| \geq 4$ for all $n > 2$.*

Proof. Without loss of generality, index $P^{(0)} = \left\{p_i^{(0)}\right\}_{i \in J}$ for $J = \{1, 2, 3, 4\}$.

For fixed $i \in J$, the point $f_i = \frac{1}{4}(3p_i^{(0)} + 2p_{i+1}^{(0)} + 2p_{i+2}^{(0)} + p_{i+3}^{(0)}) \in P^{(2)}$, taking indices modulo 4, has multiplicity at least 3 and is fixed. To see three distinct constructions of this point from the original $p_i^{(0)}$ by midpoints, consider

$$\begin{aligned}
&\frac{1}{4}\left(3p_1^{(0)} + 2p_2^{(0)} + 2p_3^{(0)} + p_4^{(0)}\right) \\
&= \frac{1}{2}\left(\frac{1}{2}(p_1^{(0)} + p_2^{(0)}) + \frac{1}{2}(p_1^{(0)} + p_3^{(0)})) + \frac{1}{2}(\frac{1}{2}(p_1^{(0)} + p_2^{(0)}) + \frac{1}{2}(p_3^{(0)} + p_4^{(0)})\right) \\
&= \frac{1}{2}\left(\frac{1}{2}(p_1^{(0)} + p_2^{(0)}) + \frac{1}{2}(p_1^{(0)} + p_4^{(0)})) + \frac{1}{2}(\frac{1}{2}(p_1^{(0)} + p_3^{(0)}) + \frac{1}{2}(p_2^{(0)} + p_3^{(0)})\right) \\
&= \frac{1}{2}\left(\frac{1}{2}(p_1^{(0)} + p_2^{(0)}) + \frac{1}{2}(p_2^{(0)} + p_3^{(0)})) + \frac{1}{2}(\frac{1}{2}(p_1^{(0)} + p_3^{(0)}) + \frac{1}{2}(p_1^{(0)} + p_4^{(0)})\right)
\end{aligned}$$

Ranging i in J, we have 4 distinct points in $F^{(3)}$, so $|F^{(3)}| \geq 4$, and by the persistence of fixed points $F^{(n)} \geq 4$ for all $n \geq 3$. □

Corollary 5.7. *If $|P^{(0)}| > 4$, then $|F^{(n)}| \geq 4$ for all $n > 2$.*

Proof. If $A = \left\{p_i^{(0)}\right\}_{i \in J} \subset P^{(0)}$ has a cardinality of 4, then by the previous proposition, the GMI on $P^{(0)}$ will produce four fixed points $f_1, \dots, f_4$ associated with A □

6. Hausdorff convergence results for the GMI

We now prove several convergence results involving the sets of fixed points $F^{(n)}$, iteration midpoints $P^{(n)}$, and their convex hulls $c(F^{(n)})$ and $c(P^{(n)})$ respectively. In doing so, we justify our earlier observation that the GMI converges to a region of positive area and that the GMI becomes dense in this region.

We observe that the sequence of convex hulls $\left\{c(P^{(n)})\right\}_{n \in \mathbb{N}}$ form a nested sequence of compact sets.

Proposition 6.1. $\lim c(F^{(n)})$ *exists, with respect to the Hausdorff metric.*

Proof. By the definition of fixed points, $F^{(n)} \subset F^{(n+1)}$. This implies the convex hulls $c(F^{(n)}) \subset c(F^{(n)})$ form an increasing sequence of subsets. Lastly, $c(F^{(n)}) \subset c(P^{(0)})$ for all n. So $\left\{c(F^{(n)})\right\}$ satisfies the conditions of Proposition 3.9. □

Proposition 6.2. $\lim d_H(F^{(n)}, c(F^{(n)})) = 0$.

Proof. Let $x \in c(F^{(k)})$ for some k. Then, by Proposition 5.6, there exist three non-collinear points

$$\{f_1, f_2, f_3\} \subset F^{(k)}$$

such that x is in their convex hull. The midpoint iteration subdivides $F^{(k)}$ into four congruent triangles. Inductively, n iterations of the midpoint procedure subdivides $F^{(k)}$ into 4^n congruent triangles.

Let T_n represent a triangle of the nth subdivision containing x. By Proposition 3.1, for arbitrary $\varepsilon > 0$, there exists an N such that for $m > N$, the vertices $\{v_1, v_2, v_3\}$ of T_m are within ε distance of the centroid c of T_m. Then

$$\inf_i d(x, v_i) \leq \sup_j d(c, v_j) < \varepsilon.$$

This implies

$$d_H(F^{(n)}, c(F^{(n)})) = \sup_{x \in c(F^{(k)})} \inf_{y \in F^{(k)}} d(x, y) < \varepsilon.$$

□

Proposition 6.3. $\lim d_H(P^{(n)}, F^{(n)}) = 0$.

Proof. For notation, let

$$d(x, A) = \inf_{y \in A} d(x, y).$$

Since $F^{(n)} \subset P^{(n)}$, the Hausdorff distance is equal to

$$d_H(P^{(n)}, F^{(n)}) = \sup_{x \in P^{(n)}} d(x, F^{(n)}).$$

Let $p' \in P^{(n+1)}$ arbitrarily. By definition of the GMI, there exists at least one pair $p, q \in P^{(n)}$ such that $\frac{p+q}{2} = p'$. Then, by comparing lengths of similar triangles created by connecting relevant points,

$$d(p', F^{(n+1)}) \leq \frac{1}{2} d(p, F^{(n)}) \leq \frac{1}{2} \sup_{x \in P^{(n)}} \inf_{y \in F^{(n)}} d(x, y).$$

Since p' was arbitrary and $P^{(n)}$ and $F^{(n)}$ are finite sets, we may choose p' such that

$$\sup_{z \in P^{(n+1)}} d(z, F^{(n+1)}) = d(p', F^{(n+1)}).$$

This is precisely the inequality

$$d_H(P^{(n+1)}, F^{(n+1)}) \leq \frac{1}{2} d_H(P^{(n)}, F^{(n)}).$$

□

Proposition 6.4. $\lim d_H(P^{(k)}, c(F^{(k)})) = 0.$

Proof. By the triangle inequality,

$$d_H(P^{(k)}, c(F^{(k)})) \leq d_H(P^{(k)}, F^{(k)}) + d_H(F^{(k)}, c(F^{(k)})).$$

The claim then follows from the previous Propositions 6.2 and 6.3. □

Theorem 6.5. *The sequence of convex hulls of fixed points $F^{(n)}$ and the sequence of convex hulls of $P^{(n)}$ have the same limit. That is,*

$$\lim c(F^{(n)}) = \lim c(P^{(n)}) = \bigcap c(P^{(n)}).$$

And consequently $\lim d_H(P^{(n)}, c(P^{(n)})) = 0$.

Proof. For notation, let $\mathcal{F} = \lim c(F^{(n)})$ and $\mathcal{P} = \lim c(P^{(n)})$. Seeking contradiction, suppose $\mathcal{F} \subsetneq \mathcal{P}$. Then $d_H(\mathcal{F}, \mathcal{P}) \neq 0$.

By Proposition 6.2, $\left\{c(F^{(n)})\right\}_{n \in \mathbb{N}}$ is also a Cauchy sequence.

By compactness, choose $p \in c(P^{(N)})$ and $f \in c(F^{(N)})$ such that

$$d(p, f) = d_H(c(P^{(N)}), c(F^{(N)})).$$

Since $\mathcal{F} \subsetneq \mathcal{P}$, $d(p, f)$ is necessarily positive. Now, the midpoint of f and p is in $F^{(N+1)}$ by Proposition 5.5. Denote $g = \frac{p+f}{2}$. Then

$$d_H(c(F^{(N)}), c(F^{(N+1)})) \geq \inf_{y \in c(F^{(N)})} d(g, y) \geq \frac{1}{2} d(p, f) > 0.$$

Passing to the limit,

$$\lim_N d_H(c(F^{(N)}), c(F^{(N+1)})) > 0,$$

contradicting the fact that $\left\{c(F^{(k)})\right\}$ is a Cauchy sequence. Therefore we conclude $\mathcal{F} = \mathcal{P}$.

Consequently,

$$d_H(P^{(n)}, c(P^{(n)})) \leq d_H(P^{(n)}, c(F^{(n)})) + d_H(c(F^{(n)}), c(P^{(n)}))$$

and

$$\lim d_H(P^{(n)}, c(P^{(n)})) = 0.$$

□

We note that Proposition 6.3 establishes that the GMI converges to its fixed points, certainly a set of positive area. Moreover, Theorem 6.5 establishes that the GMI becomes dense in itself because the points themselves limit to their convex hull.

7. Centroid iteration for polyhedra

The midpoint iteration yields another generalization, the centroid iteration for polyhedra. (We could also call this the derived polyhedron iteration.) The centroid iteration takes a polyhedron and forms a new polyhedron by taking the centroid of every face to be a new vertex, which is analogous to taking the midpoint of each side of a polygon.

Experimentally, the centroid iteration shares properties with the GMI in that it usually limits to a region of positive volume and that it becomes dense in the boundary of this region. It may also share the property of regularity in the limit with the midpoint iteration, as simulations suggest that polyhedra limit to ellipsoidal shapes. Unfortunately, simulating this process is very computationally expensive.

More work is required to see if the Hausdorff convergence results for the GMI will suggest proofs for convergence results for the polyhedron centroid iteration.

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