UNIFORM CONVERGENCE OF MIDPOINT CURVE SEQUENCES

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Abstract. Let f be a continuous piecewise linear plane curve parameterized by the closed interval $I = [0, 1]$. We define a midpoint iteration on f in $C(I, \mathbb{R}^2)$ and obtain convergence results in the d_∞ metric for the case when f has three vertices.

1. Introduction

Let I denote the closed interval $[0, 1]$. Let $C(I, \mathbb{R}^2)$ be the set of all continuous functions from I to $\mathbb{R}^2$. The function $d_\infty(f, g) = \sup_{t \in I} |f(t) - g(t)|$ gives $C(I, \mathbb{R}^2)$ the structure of a Banach space.

Let a and b be points of the real plane $\mathbb{R}^2$. Then denote by the *line segment* L_a^b the element of $C(I, \mathbb{R}^2)$ with starting point a and endpoint b given by:

$$L_a^b(t) = tb + (1 - t)a$$

A function $f \in C(I, \mathbb{R}^2)$ is a *piecewise linear curve* if it is a piecewise function whose parts are line segments. It will be convenient to refer to the set $V(f)$ of vertices of f, where we adopt the convention of enumerating the vertices in the order that f passes through them, starting with $f(0)$ and ending with $f(1)$.

Given a continuous piecewise linear curve, we define a sequence of continuous piecewise linear curves in the following way.

Definition 1.1. Let $f \in C(I, \mathbb{R}^2)$ be a piecewise linear curve. Inductively define $f_0 = f$. Then, for subsequent n in $\mathbb{N}$, let

$$V_n = \left\{ \frac{1}{2}(v_i + v_{i+1}) : v_i \in V_{n-1} \right\} \cup \{f(0), f(1)\}$$

and define f_n as the continuous piecewise linear function which connects the vertices in V_n in the natural order, starting from $f(0)$, to $\frac{1}{2}(f(0) + v_1)$, and ending with $f(1)$. Call $\{f_n\}_{n \in \mathbb{N}}$ the *midpoint sequence* of f.

In the notation above, the set $V_n = V(f_n)$. We illustrate this procedure with an example.

Example: Let f be the curve which connects the points $(0, 1)$, $(0, 0)$, and $(1, 0)$ of $\mathbb{R}^2$, in that order. Then f_1 has the vertex set:

$$V(f_1) = \{(0, 1), (0, 1/2), (1/2, 0), (1, 0)\}$$

We will refer to this example curve as the *right angle curve*. Elements of the midpoint sequence of the right angle curve are shown in Figure 1.

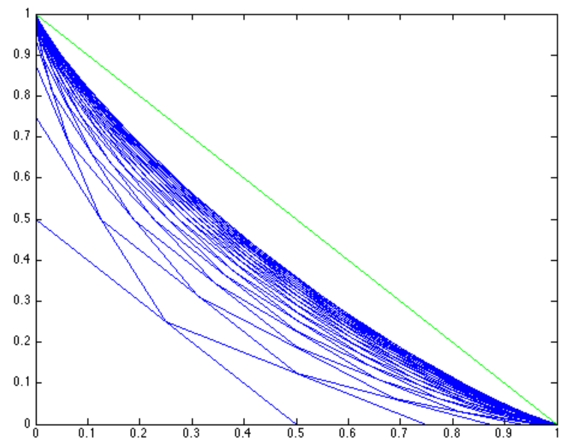

FIGURE 1. The midpoint sequence of the right angle curve

The image above suggests the possibility of convergence in $C(I, \mathbb{R}^2)$ of this midpoint sequence to the line segment connecting $(0, 1)$ to $(1, 0)$, shown in green. Denote this line segment by $L(t)$. We will establish this convergence with our main result:

Theorem 2.3 *The midpoint sequence $\{f_n\}_{n\in\mathbb{N}}$ of the right angle curve converges with respect to the d_∞ metric of $C(I, \mathbb{R}^2)$ to $L(t)$.*

Theorem 2.3 will have an application towards O'Rourke's problem [1], which can be stated for subsets of $\mathbb{R}^2$ as follows.

Problem 1.2. *Let $V_0 \subset \mathbb{R}^2$ be a finite set of vertices, and $c(V_0)$ its convex hull. Inductively define a sequence of nonempty subsets:*

$$V_n = \left\{ \frac{1}{2}(v + w) : v, w \in V_{n-1} \text{ and } v \neq w \right\}$$

That is, each V_n is the set of midpoints of all possible pairs of distinct points in V_{n-1}. What is the limit of the corresponding sequence $\{c(V_n)\}_{n\in\mathbb{N}}$ of convex hulls?

The fact that $\{c(V_n)\}_{n\in\mathbb{N}}$ is a sequence of compact, nonempty and nested subsets of $\mathbb{R}^2$ implies the limit exists with respect to the Hausdorff metric [2]. We will use the midpoint sequence of the right angle curve to study the topological boundary of this limit in the case that V_0 is the set of vertices of the unit square.

2. Convergence of the Right Angle Curve Sequence

In this section, let f be the right angle curve, defined previously, and $\{f_n\}_{n\in\mathbb{N}}$ its midpoint sequence. Denote $L(t) = L_{(0,1)}^{(1,0)}(t)$ for brevity. To show $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to $L(t)$, we will need a proposition concerning the limit of sums of binomial coefficients.

Proposition 2.1. *Let* $P_{2n} = \frac{1}{2^{2n}} \sum_{j=0}^{n-1} \binom{2n}{j}$*, the partial sum up to* $n-1$ *of the* $2n$*th row of binomial coefficients. Then* $\lim P_{2n} = \frac{1}{2}$.

Proof. By Pascal's relation:

$$2^{2n} = \sum_{j=0}^{2n} \binom{2n}{j} = 2\sum_{j=0}^{n-1} \binom{2n}{j} + \binom{2n}{n}$$

We can rewrite this equality as:

$$\frac{2^{2n-1} - \frac{1}{2}\binom{2n}{n}}{2^{2n}} = \frac{1}{2^{2n}} \sum_{j=0}^{n} \binom{2n}{j} = P_{2n}$$

By Stirling's approximation, for sufficiently large n

$$\frac{1}{2^{2n+1}}\binom{2n}{n} = \frac{1}{2^{2n+1}}(1+\varepsilon_n)(\frac{2^{2n}}{\sqrt{\pi n}})$$

where $\varepsilon_n > 0$ is a small constant depending on n such that $\lim \varepsilon_n = 0$. From here, we can conclude

$$\lim \frac{1}{2^{2n+1}}\binom{2n}{n} = \lim \frac{1+\varepsilon_n}{2\sqrt{\pi n}} = 0$$

Using this limit result in the limit of P_{2n} yields

$$\lim P_{2n} = \frac{1}{2} - \lim \frac{1}{2^{2n+1}}\binom{2n}{n} = \frac{1}{2}$$

□

Note that if $k = 2n$ is even, then there are an odd number of vertices in $V(f_k)$. Furthermore, there exists exactly one "symmetric" vertex of the form $(\frac{a}{2^{2n}}, \frac{a}{2^{2n}})$ for some natural number a. For example, if $k = 2$:

$$V(f_2) = \{(0,1), (0,3/4), (1/4,1/4), (3/4,0), (1,0)\}$$

and the vertex $(1/4, 1/4)$ is symmetric in the x and y coordinates. The distance from $(1/4, 1/4)$ represents $d_\infty(f_2, L)$.

We will establish that the numerator of the symmetric vertex of $V(f_{2n})$ is a partial sum of binomial coefficients, leading to our convergence result.

Proposition 2.2. *For all* $n \in \mathbb{N}$*, the vertex set* $V(f_{2n})$ *contains a single symmetric vertex of the form* (P_{2n}, P_{2n})*, where:*

$$P_{2n} = \frac{1}{2^{2n}} \sum_{j=0}^{n-1} \binom{2n}{j}$$

Proof. For convention, enumerate the vertices of $V(f_{2n}) = \{v_1, \ldots, v_{2n+3}\}$ in the order that f_{2n} passes through them, starting from $v_1 = (0,1)$ and ending at $(1,0)$.

The example above for $V(f_2)$ proves the case $n = 1$. We also observe that the $x-$coordinate of $(0, 3/4)$ is $P_2 - \frac{1}{2^2}\binom{2}{0}$, and the $x-$coordinate of $(3/4, 0)$ is $P_2 + \frac{1}{2^2}\binom{2}{1}$.

So suppose the proposition holds for P_{2j}, where $j < n$, and that the $x-$coordinate of the vertex v_{n+1} is $P_{2n} - \frac{1}{2^{2n}}\binom{2n}{n-1}$, and the $x-$coordinate of v_{n+3} is $P_{2n} + \frac{1}{2^{2n}}\binom{2n}{n}$. The vertex v_{n+2} is the symmetric vertex, with $x-$coordinate P_{2n}.

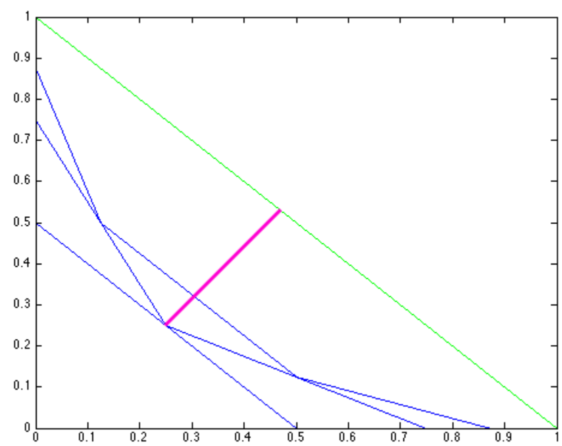

FIGURE 2. Visual representation of $d_\infty(f_2, L)$ as the purple line segment from the image of f_2 to the image of $L(t)$.

It is a straightforward (but tedious) calculation to show

$$P_{2n} + \frac{1}{2^{2n+2}}(\binom{2n}{n} - \binom{2n}{n-1}) = \frac{1}{2^{2n+2}}\sum_{j=0}^{n}\binom{2n+2}{j} = P_{2n+2}$$

But the quantity $P_{2n} + \frac{1}{2^{2n+2}}(\binom{2n}{n} - \binom{2n}{n-1})$ is the $x-$coordinate of the symmetric vertex in $V(f_{2n+2})$ by the midpoint iteration. And by symmetry of the calculations, the $y-$coordinate of the symmetric vertex in $V(f_{2n+2})$ is also P_{2n+2}. This concludes the induction. □

Note from Figure 2 that the symmetric vertex of $V(f_{2n})$ is the point on the image of f_{2n} farthest from $L(t)$, in terms of perpendicular distance. We may now prove our main result.

Theorem 2.3. *The midpoint sequence $\{f_n\}_{n\in\mathbb{N}}$ of the right angle curve converges with respect to the d_∞ metric of $C(I, \mathbb{R}^2)$ to $L(t)$.*

Proof. Let $n \in \mathbb{N}$. The line segments comprising f_{2n} determine a parameterization by $t \in [0, 1]$ such that, by proposition 3.2

$$d_\infty(f_{2n}, L) = \sup_{t\in[0,1]} |f_{2n}(t) - L(t)| = |(P_{2n}, P_{2n}) - (\frac{1}{2}, \frac{1}{2})|$$

Using proposition 3.1, we pass to the limit and see

$$\lim |(P_{2n}, P_{2n}) - (\frac{1}{2}, \frac{1}{2})| = 0$$

Now, the midpoint sequence $\{f_n\}$ is a monotone sequence with respect to the partial order on $C(I, \mathbb{R}^2)$, which is to say

$$f_n(x) \leq f_{n+1}(x) \leq f_{n+2}(x)$$

for all $x \in I$. This implies the entire sequence f_n converges uniformly to $L(t)$. □

Theorem 3.3 implies a corollary regarding continuous piecewise linear curves of any three vertices.

Corollary 2.4. *Let f be a piecewise linear curve connecting three distinct non-collinear points a, b, c of the plane, in that order, and let $\{f_n\}_{n\in\mathbb{N}}$ be its midpoint sequence. Then f_n converges in $C(I, \mathbb{R}^2)$ to $L_a^c(t)$ with respect to the d_∞ metric.*

Proof. By shifting b to the origin, we can assume a and c are the only nonzero vertices of f.

Then by modifying the argument in proposition 3.2 we establish the existence of a unique symmetric vertex v_{2n} in $V(f_{2n})$ such that

$$v_{2n} = P_{2n}(a + c)$$

v_{2n} is the furthest point on the image of f_{2n} from the image of $L_a^c(t)$. But $\lim P_{2n}(a + c) = \frac{1}{2}(a + c)$. Conclude $\lim d_\infty(f_n, L_a^c) = 0$. □

3. n-point curves and applications to O'Rourke's conjecture

As a standalone question, one may ask if a similar convergence result holds for "n-point" continuous piecewise linear plane curves. In fact, numerical experiments in `MATLAB` suggest the following conjecture for n-point curves.

Conjecture 3.1. *Let $f \in C(I, \mathbb{R}^2)$ be a piecewise linear curve connecting any n points of the plane. Let $\{f_n\}_{n\in\mathbb{N}}$ be its midpoint sequence. Then the f_n converge uniformly to a line segment $L(t)$ in $C(I, \mathbb{R}^2)$ such that $L(0) = f(0)$ and $L(1) = f(1)$.*

This plausible conjecture has an interesting geometric interpretation. Given a polygonal path on the plane, the midpoint procedure would converge to the line segment between the endpoints of the path and form a closed polygon. This result would be in the spirit of the Darboux midpoint polygon problem [3], made famous by I.J. Schoenberg in the American Mathematical Monthly. [4]

As an application, we may use Theorem 2.3 towards understanding O'Rourke's problem in the following way. In the case where V_0 is the four corners of the unit square, numerical experiments suggest that the topological boundary of the limit of the $c(V_n)$ contains no line segments. We can identify each side of V_0 with a right angle curve. Then Theorem 2.3 implies that vertices must be added infinitely many times onto the boundaries of the $c(V_n)$ during the limit process, independently from the separate midpoint procedures occurring on each corner of the boundary, since otherwise the $c(V_n)$ would necessarily converge to a set with a piecewise linear boundary.

4. Acknowledgments

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