A Reversed $\Sigma-$Shaped Bifurcation Curve for a Class of Superlinear Boundary Value Problems

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Overview

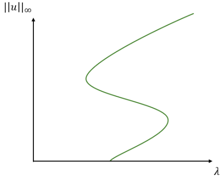
Motivation

Previous Research

T. Laetsch was the first to use a quadrature method to explore the solutions of two-point boundary value problems of the form:

$$\begin{cases} -u'' = \lambda f(u); (0,1) \\ u(0) = 0 = u(1) \end{cases}$$

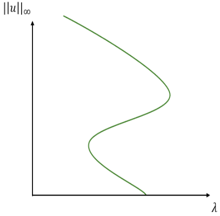
Additionally, K. Brown, M. Ibrahim, and R. Shivaji explored the conditions under which the positive solutions would form an S-shaped bifurcation curve.

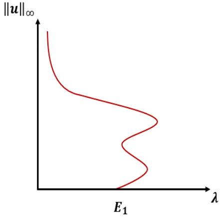


Laetsch, T. *The number of solutions of a nonlinear two point boundary value problem*, Bull. Math. Biol., vol. 81, no. 10, pp. 1-13, 1970/1971.

K. Brown, M. Ibrahim, and R. Shivaji *S-shaped bifurcation curves*, Nonlinear Anal., vol. 5, no. 5, pp. 475–486, 1981

Furthermore, S.-H. Wang and T.-S. Yeh explored the conditions under which the bifurcation curve would be a reverse S-shape. This research closely aligns with the objective of the current study.





S.-H. Wang and T.-S. Yeh, *theorem on reversed S-shaped bifurcation curves for a class of boundary value problems and its application*, Nonlinear Anal., vol. 71, no. 1-2, pp. 126–140, 200.

Model of Interest

Two-Point Boundary Problems

In this project, we explore positive solutions of two-point boundary value problems of the form:

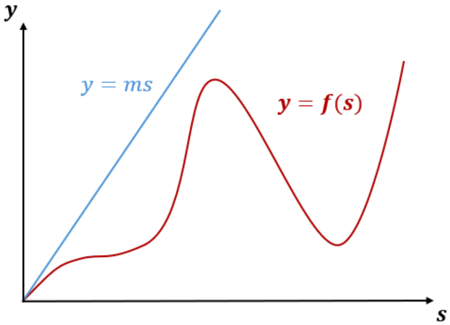
$$\begin{cases} -u'' = \lambda f(u); (0,1) \\ u(0) = 0 = u(1) \end{cases} \tag{1}$$

Here, we assume the following:

- $f(s) > 0$ for $s > 0$
- $f(0) = 0$
- f is an infinitely differentiable function
- $f'(0) = m$ for some fixed $m > 0$, and
- $\lim_{s \to \infty} \frac{f(s)}{s} = \infty$ (ie. f is superlinear at ∞)

The next slide demonstrates these assumptions

Graph Demonstrating Assumptions



Form of $f(s)$ Under Consideration

(H_1): There exist $b > a > 0$ such that $f(s) < \frac{f(a)}{a}s$ for $s \in (0, a)$ and $f(s) > \frac{f(b)}{b}s$ for $s \in (0, b)$.

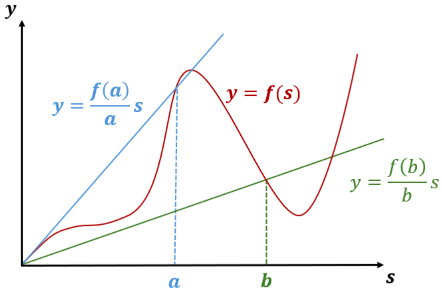
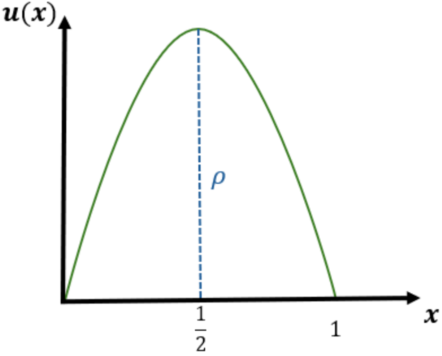


Figure: Form for $f(s)$

Shape of the Solution ($u(x)$) Profile

With $u(x)$ as a positive solution of (1) such that $||u||_\infty = \rho$, it follows by Picard's Existence and Uniqueness Theorem that $u(x)$ is symmetric about $x = \frac{1}{2}$.



Method

Method: Quadature Method (Developed by T. Laetsch)

Lemma 1

A positive solution, $u(x)$, of (1) with $u(\frac{1}{2}) = ||u||_\infty = \rho$, $u(0) = 0 = u(1)$ exists if and only if $\lambda > 0, \rho > 0$ satisfy:

$$\sqrt{\lambda} = G(\rho) = \sqrt{2} \int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \tag{2}$$

where $F(s) = \int_0^s f(t)dt$ and $G(\rho)$ is a differentiable function of ρ.

Laetsch, T. *The number of solutions of a nonlinear two point boundary value problem*, Bull. Math. Biol., vol. 81, no. 10, pp. 1-13, 1970/1971.

Formula for $\frac{d\sqrt{\lambda}}{d\rho}$

Lemma 2

It also follows from Brown, Ibrahim, and Sivaji that

$$\frac{d\sqrt{\lambda}}{d\rho} = \sqrt{2}\int_0^1 \frac{H(\rho) - H(\rho v)}{(F(\rho) - F(\rho v))^{\frac{3}{2}}}\, dv \tag{3}$$

where

$$H(s) = F(s) - \frac{sf(s)}{2}$$

This result is developed using the Lebesgue dominated convergence theorem.

K. Brown, M. Ibrahim, and R. Shivaji *S-shaped bifurcation curves*, Nonlinear Anal., vol. 5, no. 5, pp. 475–486, 1981

Key Results

Results

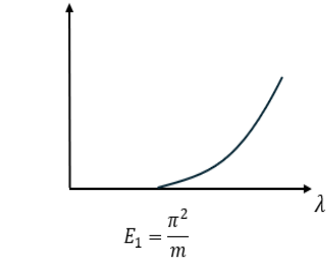
Theorem 1

If u is a positive solution of (1), then $\lambda \to \frac{\pi^2}{m} (= E_1)$ as $\rho = \|u\|_\infty \to 0$.

Here, E_1 is the principal eigenvalue of:

$$\begin{cases} -\phi'' = mE\phi; \ (0,1) \\ \phi(0) = \phi(1) = 0. \end{cases} \quad (4)$$

Equation (4) above is the linearized eigenvalue problem associated with the model.

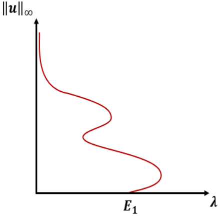


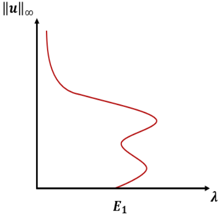
Results

Theorem 2

Let $f''(0) < 0$ and (H_1) hold. Then, the bifurcation diagram of positive solutions to (1) is roughly reversed Σ - shaped.

Expected bifurcation curves are given below.





Proof of Theorem 2

Note that this proof is based on geometric arguments.

We first show that the bifurcation curve turns to the right from E_1 when $f''(0) < 0$. To do so, it is enough to show that $\frac{d\sqrt{\lambda}}{d\rho} > 0$ for $\rho \approx 0$.

First, because $F(s)$ is increasing, this implies that $F(\rho) - F(\rho v) > 0$ for $v \in (0, 1)$. Therefore, the denominator is always positive and we must prove that $H(\rho) - H(\rho v)$ is also positive.

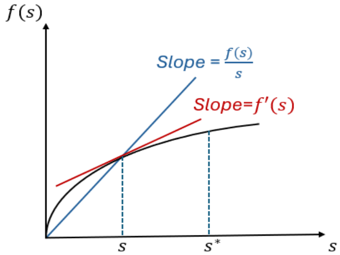
$$\frac{d\sqrt{\lambda}}{d\rho} = \sqrt{2} \int_0^1 \frac{H(\rho) - H(\rho v)}{(F(\rho) - F(\rho v))^{\frac{3}{2}}} \, dv$$

Proof of Theorem 2: Continued

By the definition of $H(s)$, $H'(s) = \frac{1}{2}[f(s) - sf'(s)] = \frac{s}{2}\left[\frac{f(s)}{s} - f'(s)\right]$

$$H(s) = F(s) - \frac{sf(s)}{2}$$

There will exist $s^* > 0$ such that $f''(s) < 0$ for $s \in (0, s^*)$. From this, we obtain the below figure.



From the figure to the left, it is clear to see that $\frac{s}{2}\left[\frac{f(s)}{s} - f'(s)\right] > 0$ for $s \in (0, s^*)$. Thus, $H(s)$ is increasing and $H(\rho) - H(\rho v) > 0$.

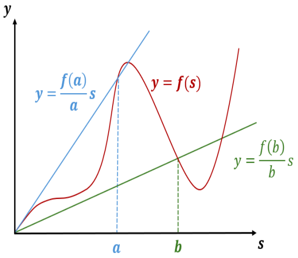
This proves that the bifurcation curve turns to the right from E_1

Proof of Theorem 2: Continued

We must now prove that $\frac{d\sqrt{\lambda}}{d\rho} < 0$ for $\rho = a$.

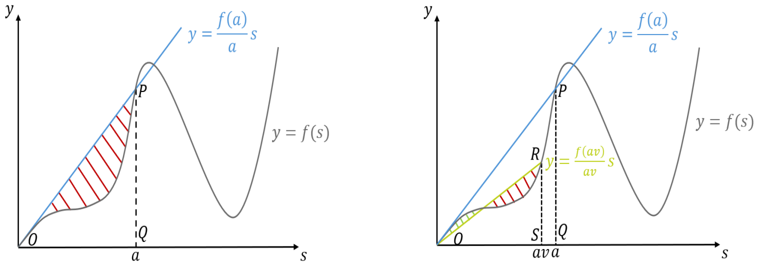
$$\frac{d\sqrt{\lambda}}{d\rho} = \sqrt{2}\int_0^1 \frac{H(\rho)-H(\rho v)}{\left(F(\rho)-F(\rho v)\right)^{\frac{3}{2}}}\,dv$$

where, $H(s) = F(s) - \frac{sf(s)}{2}$.



To prove this, we again use the equation for $\frac{d\sqrt{\lambda}}{d\rho}$ and the geometry of the $f(s)$.

Proof of Theorem 2: Continued

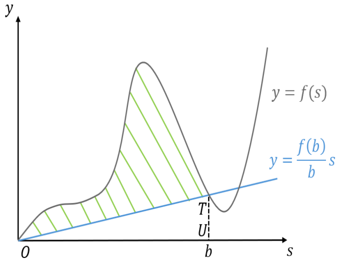


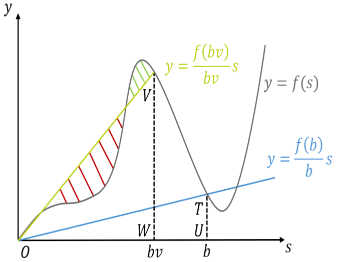
$$H(s) = F(s) - \frac{sf(s)}{2}$$

From the figure above, it is clear to see that $H(s)$ will have negative values for both figures, with $H(a)$ being less than $H(av)$. Thus, $H(a) - H(av)$ will be negative and $\frac{d\sqrt{\lambda}}{d\rho} < 0$. Therefore, the bifurcation curve will come back to the left.

Finally, we prove that $\frac{d\sqrt{\lambda}}{d\rho} > 0$ for $\rho =$ b geometrically below.

Proof of Theorem 2: Continued





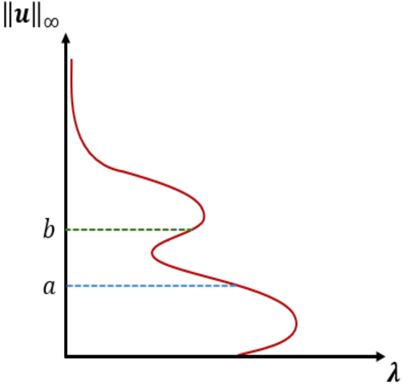
$$H(s) = F(s) - \frac{sf(s)}{2}$$

The figure above shows that $H(b)$ is positive and $H(bv)$ is negative, meaning that $H(b) - H(bv)$ will be positive. Therefore, $\frac{d\sqrt{\lambda}}{d\rho} > 0$ and the bifurcation curve will again turn to the right.

Proof of Theorem 2: Continued

The figure illustrates potential values of $\rho = a$ and $\rho = b$ on the bifurcation curve.

Also, note that it is proved by T. Laetsch that as $\lambda \to 0$, $\rho \to \infty$. This completes the proof of theorem 2.



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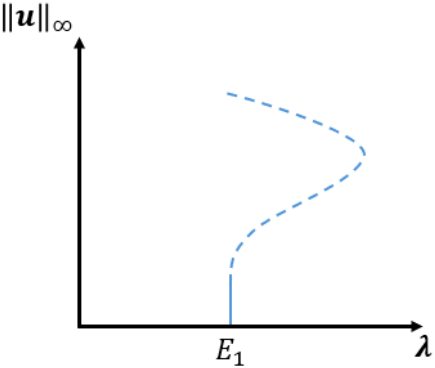
Results

Theorem 3

Let $f''(0) = 0$. Then (1) has a positive solution u such that $\lim_{\rho \to 0^+} \frac{d\sqrt{\lambda}}{d\rho} = 0$ (see Figure below).

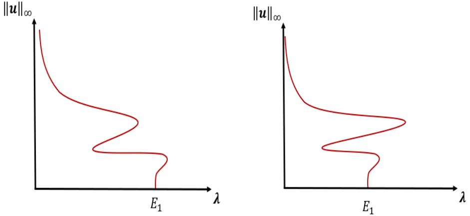
This result holds for any class of functions.



Theorem 4

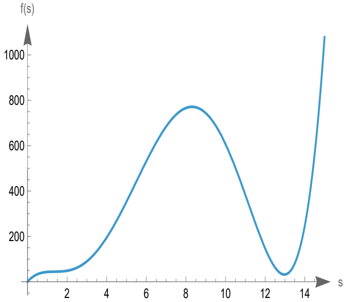
Let $f''(0) = 0$, $f''(s) < 0$ for $s \in (0, c_0)$ for some $c_0 > 0$, and (H_1) hold. Then the bifurcation diagram of positive solutions to (1) is roughly reversed Σ-shaped.

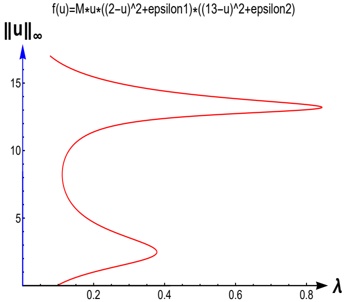
Expected bifurcation curves are shown below.



Numerical Example

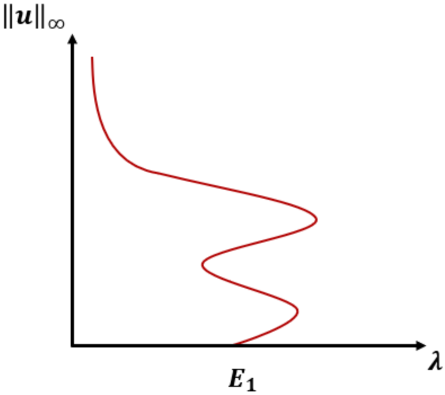
Here, I demonstrate a numerical example of f(s) and the resulting bifurcation curve. Notice that this example includes four positive solutions in the bifurcation diagram.





Future Work

In future research, we will explore conditions under which the bifurcation curve will result in four positive solutions as shown in the figure below.



Questions

Thank You!

Loren Baker