An Alternative Proof of the Newton-Girard Formula for Non-Commutative Symmetric Polynomials

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1. If $n = 3$, then $x_1^2 x_2^2 x_3^2$ is a symmetric polynomial, because

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2. If $n = 3$, then $x_1^2 x_2 x_3$ is **not** a symmetric polynomial, because

$$x_1^2 x_2 x_3 \neq x_2^2 x_1 x_3 \neq x_3^2 x_1 x_2.$$

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Examples:

1. If $n = 3$, then $p_3 = x_1^3 + x_2^3 + x_3^3$
2. If $n = 5$, then $p_6 = x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6$

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3. If $n = 5,\ k = 5$: $e_5 = x_1x_2x_3x_4x_5$

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Note: e_k is only defined for $k \leq n$

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$$\begin{aligned}\sum_{j=1}^{2}(-1)^{j-1}p_je_{2-j} &= p_1e_{2-1} - p_2e_{2-2} \\ &= p_1e_1 - p_2 \\ &= (x_1+x_2+x_3)(x_1+x_2+x_3) - (x_1^2+x_2^2+x_3^2)\end{aligned}$$

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&= x_1^2 + x_1x_2 + x_1x_3 + x_1x_2 + x_2^2 + x_2x_3 \\
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&= 2x_1x_2 + 2x_1x_3 + 2x_2x_3 \\
\therefore \quad LHS &= RHS
\end{aligned}$$

Partitions

Definition.

A **partition** of a non-negative integer m is a sequence of non-negative integers in non-increasing order that sum to m, which contains only finitely many zero terms.

Here are some partitions of 2:

$$(2),\ (1,1),\ (2,0),\ (1,1,0),\ (2,0,0)$$

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Given a partition λ we define the **length**, $\ell(\lambda)$ to be the number of parts of λ. We use powers to count the multiplicity of elements in a partition so that $(1^2) = (1, 1)$.

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1. $\ell\left(2,1^2,0\right) = \ell\left(2,1,1,0\right) = 3$
2. $\ell\left(5^2,4,2^4,1,0^2\right) = \ell\left(5,5,4,2,2,2,2,1,0,0\right) = 8$

Monomial Symmetric Polynomials

Definition.

Let λ be a partition with $\ell(\lambda) \leq n$. Adding zeros if $\ell(\lambda) < n$, write $\lambda = (b_1, \ldots, b_n)$. Then define the **monomial symmetric polynomial** given by λ in n variables by

$$m_\lambda = \sum x_1^{c_1} x_2^{c_2} \cdots x_n^{c_n}.$$

Where the sum is over all distinct permutations $(c_1, \ldots, c_n)$ of $(b_1, \ldots, b_n)$. If $\ell(\lambda) > n$ we define $m_\lambda := 0$.

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Let $n = 3$. Then:

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4. $m_{(2^2,1^2)} = 0$

Note:

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Thus:

Monomial symmetric polynomials are generalizations of the power sum and elementary symmetric polynomials.

Lemma 1: (Mead)

Let k be a positive integer with $k \leq n$. Then for all $i \in \{2, 3, \ldots, k-1\}$,

$$p_i e_{k-i} = m_{(i+1, 1^{k-i-1})} + m_{(i, 1^{k-i})}$$

And

$$p_1 e_{k-1} = m_{(2, 1^{k-2})} + k e_k$$

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Let $k = 3, n = 3$, and $i = 2$, then

$$\begin{aligned} LHS &= p_2 e_{3-2} \\ &= p_2 e_1 \\ &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3) \\ &= x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_1 + x_2^3 + x_2^2 x_3 \\ &\quad + x_3^2 x_1 + x_3^2 x_2 + x_3^3 \end{aligned}$$

Mead's Lemma

$$LHS = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^3 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 + x_3^3$$

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From **Lemma 1**:

$$p_i e_{k-i} = m_{(i+1,1^{k-i-1})} + m_{(i,1^{k-i})}$$

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In a non-commutative setting:

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Definition.

The symmetric group S_n is the set of all bijective functions from the set $\{1, 2, \ldots, n\}$ to itself. An element of S_n is called a **permutation**.

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We will use **cycle notation** to denote permutations.

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The permutation $(123) \in S_3$ in cycle notation corresponds to the bijective function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ defined by $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$.

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We denote the identity permutation (the one that changes nothing) by (1). Also, any number that is not present in a cycle is fixed. So $(12) \in S_3$ fixes 3.

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The inverse of (12) is denoted by $(12)^{-1} = (21) = (12)$
The inverse of (123) is denoted by $(123)^{-1} = (321) = (132)$

Non-Commutative Elementary Sum Polynomial

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The non-commutative elementary polynomial of degree $k \leq n$ is

$$E_k := \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(k)}$$

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$$E_2 = x_1 x_2 + x_2 x_1 + x_3 x_2 + x_1 x_3 + x_2 x_3 + x_3 x_2$$

Shuffle Sets

Definition.

Given $k \leq n$ and $i \in \{0, 1, \ldots, k\}$, we define Sh_i the set of all **shuffles** to be the set of all $\sigma \in S_k$ with the property that σ^{-1} preserves the orders both of $1, 2, \ldots, k-i$ and of $k-i+1, k-i+2, \ldots, k$.

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6. $(132)^{-1} = (123)$ sends 1 to 2 and 2 to 3. Thus $(132) \in Sh_1$.

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$$\therefore \quad Sh_1 = \{(1), (23), (132)\}.$$

Definition.

We define the action of the symmetric group S_k on the right on a symmetric polynomial of degree k by the rule

$$(x_{i_1} x_{i_2} \cdots x_{i_k}) \circ \sigma := x_{i_{\sigma^{-1}(1)}} x_{i_{\sigma^{-1}(2)}} \cdots x_{i_{\sigma^{-1}(k)}}$$

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Example:
$(12)^{-1} = (12)$. So, if $k = 2$ and $n = 3$, then

$$\begin{aligned}
E_2 \circ (12) &= (x_1x_2 + x_1x_3 + x_2x_1 + x_2x_3 + x_3x_1 + x_3x_2) \circ (12) \\
&= (x_1x_2) \circ (12) + (x_1x_3) \circ (12) + (x_2x_1) \circ (12) \\
&\quad + (x_2x_3) \circ (12) + (x_3x_1) \circ (12) + (x_3x_2) \circ (12) \\
&= x_2x_1 + x_3x_1 + x_1x_2 + x_3x_2 + x_1x_3 + x_2x_3
\end{aligned}$$

BDDK Theorem
(Boumova, Drensky, Dzhundrekov, Kassabov 2022)

If $k \leq n$, then

$$kE_k = (-1)^{k+1} k! p_k + \sum_{i=1}^{k-1} (-1)^{i+1} i! \left(E_{k-i} p_i \circ \sum_{\sigma \in Sh_i} \sigma \right)$$

Lemma.

For $k \leq n$. When $i > 1$:

$$E_{k-i}p_i \circ \sum_{\sigma \in Sh_i} \sigma = (i+1)\left(M_{(i+1,1^{k-i-1})} \circ \sum_{\sigma \in Sh_{k-i-1}} \sigma\right) + M_{(i,1^{k-i})} \circ \sum_{\sigma \in Sh_{k-i}} \sigma$$

When $i = 1$:

$$E_{k-1}p_1 \circ \sum_{\sigma \in Sh_1} \sigma = 2\left(M_{(2,1^{k-2})} \circ \sum_{\sigma \in Sh_{k-2}} \sigma\right) + kM_{(1^k)}$$

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We are still working on developing a lemma for the $k > n$ case of the non-commutative BDDK formula.

This has not been finalized yet and will require further work.