Where Cars Meet Combinatorics

A LaTeX-powered Presentation on Parking Functions

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The Parking Problem [1]

Imagine that we are looking at a one-way street with a dead end, and there is parking available on only one side of the street. There are n parking spots available on the street, and they are labeled 1 to n from the start of the street to the end in consecutive order. There is a line of n cars waiting to park.

Each of these drivers has a favorite parking spot; we call each driver's favorite spot their **preference**. Note that each preference is not required to be distinct from the others; multiple drivers can have the same favorite spot. In order to keep track of the parking preferences of each driver, we list them in a vector in the same order that the cars are lined up to park; that is, the ith car in line will have its preference listed in the ith spot of this vector, which we call a **preference vector.** Preference vectors allow us to easily document the cars' favorite spots.

The Parking Problem [1]

In this scenario, cars park according to the following **parking rule**: Cars enter the street in the order they are lined up. Each car drives to its preferred parking spot and checks if it is available.

- If the spot is empty, the car parks in their favorite spot - hooray! Then the next car would enter the street and repeat the process.
- If instead the car had found its preference occupied, then the car continues down the street and parks in the next available spot.

This process continues until a car reaches the dead end and cannot park or all cars are able to park. If a preference vector allows all cars to park using this parking rule, then it is called a **parking function**.

Examples of Preference Vectors [1]

Consider the preference vector $(1, 1, 4, 2)$.

1. Car 1 goes to its preference, spot 1, and parks there.
2. Car 2 goes to its preference, spot 1, and finds it occupied. It then parks in the next available spot, spot 2.
3. Car 3 goes to its preference, spot 4, and parks there.
4. Car 4 goes to its preference, spot 2, and finds it occupied. It then parks in the next available spot, spot 3.

Thus, the preference vector $(1, 1, 4, 2)$ is a parking function, since all cars were able to park on the street.

Examples of Preference Vectors [1]

Consider the preference vector $(2, 2, 3, 4)$. Although the first three cars can park in spots 2, 3, and 4 respectively, the fourth car hits the dead end and cannot park.

Thus, the preference vector $(2, 2, 3, 4)$ is not a parking function, since not all cars were able to park on the street.

Preference Vectors and Parking Functions of $n = 3$ [1]

There are 27 possible preference vectors in $n = 3$, listed below.

$$
\begin{matrix}
(1,1,1) & (1,1,2) & (1,1,3) & (1,2,1) & (1,2,2) & (1,2,3) & (1,3,1) & (1,3,2) & (1,3,3) \\
(2,1,1) & (2,1,2) & (2,1,3) & (2,2,1) & (2,2,2) & (2,2,3) & (2,3,1) & (2,3,2) & (2,3,3) \\
(3,1,1) & (3,1,2) & (3,1,3) & (3,2,1) & (3,2,2) & (3,2,3) & (3,3,1) & (3,3,2) & (3,3,3)
\end{matrix}
$$

16 of these 27 preference vectors are parking functions.

$$
\begin{matrix}
(1,1,1) & (1,1,2) & (1,1,3) & (1,2,1) & (1,2,2) & (1,2,3) & (1,3,1) & (1,3,2) \\
(2,1,1) & (2,1,2) & (2,1,3) & (2,2,1) & (2,3,1) & (3,1,1) & (3,1,2) & (3,2,1)
\end{matrix}
$$

Counting Parking Functions [1]

The number of possible preference vectors for a given n is n^n. This is evident for $n = 3$, where $3^3 = 27$. This is simple combinatorics, where n cars get to choose from n preferences, and those preferences can repeat.

The number of parking functions for a given n, or $|PF_n|$, is equivalent to $(n+1)^{n-1}$. We can see this to be true with $n = 3$, as $4^2 = 16$. The proof for this simple formula is a bit more extensive...

Proof of $|PF_n| = (n+1)^{n-1}$ [2]

Sentence 1: Add an additional space $n+1$ and arrange the spaces in a circle.

Sentence 2: Allow $n+1$ also as a preferred space.

Sentence 3: Now all cars can park, and there will be one empty space.

Proof of $|PF_n| = (n+1)^{n-1}$, with $n = 3$ as Example [2]

Sentence 1: Add an additional space $n+1$ and arrange the spaces in a circle.
Sentence 2: Allow $n+1$ also as a preferred space.
Sentence 3: Now all cars can park, and there will be one empty space.

In $n = 3$, we start with a line of three spots for the three cars to park in.

1. Our line of three spots changes into a circle with four spots and one entry point.
2. Cars can choose spot 4 as their preference.
3. All three cars will be able to park and there will be one spot left.

Proof of $|PF_n| = (n+1)^{n-1}$ [2]

Sentence 4: α is a parking function if and only if the empty space is $n+1$.
Consider $\alpha = (a_1, ..., a_n)$. This is new notation: α is an arbitrary preference vector, and each a_i for $i = 1, ..., n$ represents the preferred spot of the ith car. Earlier, we mentioned that α is a parking function if and only if it allows all of the cars to park on a street with n spots available. Thus, if we add an $(n+1)$st spot at the end of the street, this new spot must be the empty spot if α is to be a parking function. A car parking in the $(n+1)$st spot of the circle means that car could not have parked on the street with n spots; thus, α would not be a parking function.

Proof of $|PF_n| = (n+1)^{n-1}$, with $n = 3$ as Example [2]

Sentence 4: α is a parking function if and only if the empty space is $n+1$.

4. α is only a parking function if the empty space is spot 4. This means that all cars must prefer one of the first three spots *and* be able to park within those spots for α to be a parking function on the original three linear spots. Further, in the other direction, if the empty space is spot 4, we know that α is a parking function.

Proof of $|PF_n| = (n+1)^{n-1}$ [2]

Sentence 5: If $\alpha = (a_1, \ldots, a_n)$ leads to car c_i parking at space p_i, then $(a_1 + j, \ldots, a_n + j)$ (modulo $n+1$) will lead to car c_i parking at space $p_i + j$ (modulo $n+1$).

Let's define some terms. The space car c_i wants to park in is a_i, but the space c_i actually parks in is p_i. Further, let j be an integer such that $0 \leq j \leq n$. We could allow j to be $n+1$, but in modulo $n+1$, $n+1 \equiv 0$ and 0 is simpler to work with. Thus, we can refer to spot $n+1$ as spot 0.

Lemma: If α is a parking function on a circular street with $n+1$ parking spots and cars park on this street in order $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ where all p_i are distinct, then $\alpha + 1 = (a_1 + 1, a_2 + 1, \ldots, a_n + 1)$(modulo $n+1$) results in the cars parking in order $\mathbf{p} = (p_1 + 1, p_2 + 1, \ldots, p_n + 1)$. This can be proven by contradiction. The same is true for all values of j, even if they are not 1.

Proof of $|PF_n| = (n+1)^{n-1}$, with $n = 3$ as Example [2]

Sentence 5: If α leads to car c_i parking at space p_i, then $(a_1 + j, ..., a_n + j)$ (modulo $n+1$) will lead to car c_i parking at space $p_i + j$ (modulo $n+1$).

Consider preference vector $(1, 2, 2)$. The following is true about $(1, 2, 2)$:

c_i	a_i	p_i
1	1	1
2	2	2
3	2	3

Now, we have to shift each a_i and p_i by j, within modulo 4 (since we are in $n = 3$).

c_i	a_i	p_i	$a_i + 1$	$p_i + 1$	$a_i + 2$	$p_i + 2$	$a_i + 3$	$p_i + 3$
1	1	1	2	2	3	3	0	0
2	2	2	3	3	0	0	1	1
3	2	3	3	0	0	1	1	2

Proof of $|PF_n| = (n+1)^{n-1}$ [2]

Sentence 6: Hence, exactly one of the vectors $(a_1 + k, a_2 + k, ..., a_n + k)$ (modulo $n+1$) is a parking function, so $f(n) = (n+1)^n/(n+1) = (n+1)^{n-1}$.

Studying a single preference vector and the resulting shifted vectors, we find that, for this set of $n+1$ preference vectors, only one is a parking function. This is because all but one of them has a car c_i such that $p_i + j \equiv 0$ (modulo $n+1$); this is problematic because we mentioned earlier that we need the 0th parking spot (the $(n+1)$st spot) to be empty in order for α to be a parking function. Recall that there are $(n+1)^n$ distinct possible preference vectors α, since there are n cars choosing from $n+1$ preference options with replacement. Each group of $n+1$ vectors consists of an α, a starting vector, and all vectors generated by $\alpha + j$ where $0 \leq j \leq n$. Yet, only one of these vectors in each group of $n+1$ vectors is a parking function, as established. Thus, we can divide the total number of preference vectors by $n+1$ to select these parking functions, which is equal to $(n+1)^n/(n+1) = (n+1)^{n-1}$.

Proof of $|PF_n| = (n+1)^{n-1}$, with $n = 3$ as Example [2]

Sentence 6: Hence, exactly one of the vectors $(a_1 + k, a_2 + k, \ldots, a_n + k)$ **(modulo** $n+1$**) is a parking function, so** $f(n) = (n+1)^n/(n+1) = (n+1)^{n-1}$.

Let's review our table from Sentence 5.

c_i	a_i	p_i	$a_i + 1$	$p_i + 1$	$a_i + 2$	$p_i + 2$	$a_i + 3$	$p_i + 3$
1	1	1	2	2	3	3	0	0
2	2	2	3	3	0	0	1	1
3	2	3	3	0	0	1	1	2

Only one of our shifted vectors, $(1, 2, 2)$, is a parking function. We know that there are $(n+1)^n$ preference vectors which is $4^3 = 64$ for $n = 3$. If we divide by the groups of $n+1$ where only one is a parking function, we get $64/4 = 16$. That is the answer we got when we counted the parking functions of $n = 3$ earlier!

Extensions of the Parking Problem [3, 4]

There are a variety of alterations that increase the complexity of the parking problem.

- What if when a car's preference was taken, they could back up and check one spot before their taken preference?
- What if a car could teleport k spots back if its preference was taken?
- What if cars were randomly assigned the capacity to back up and check the previous spot or the obligation to move forward after checking their preference? What would be the probability that a preference vector is a parking function?
- What if cars are different lengths and need more than one consecutive spot to park?
- What if you had cars parked in all spots, but individuals had preferred cars to take a trip in? What would this "passenger problem" look like?

Bibliography

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[4] R. Ehrenbord and A. Happ, Parking Cars of Different Sizes, *The American Mathematical Monthly* (10) **123** (2016), 1045-1048.