VC-dimension of subsets of Hamming graphs

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Contents

VC-dimension

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The *VC-dimension* of the pair $(X, \mathcal{F})$ is the size of the largest subset $W \subset X$ that can be shattered by $\mathcal{F}$.

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Shattering

Given $W \subseteq X$, we say $\mathcal{F}$ *shatters* W if for every subset $S \subseteq W$, there exists an $F_S \in \mathcal{F}$ such that $W \cap F_S = S$.

Hamming graph

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The *Hamming graph* $H(d, q, t)$ has vertex set $\mathbb{Z}_q^d$, and vertices are adjacent when they have Hamming distance exactly t.

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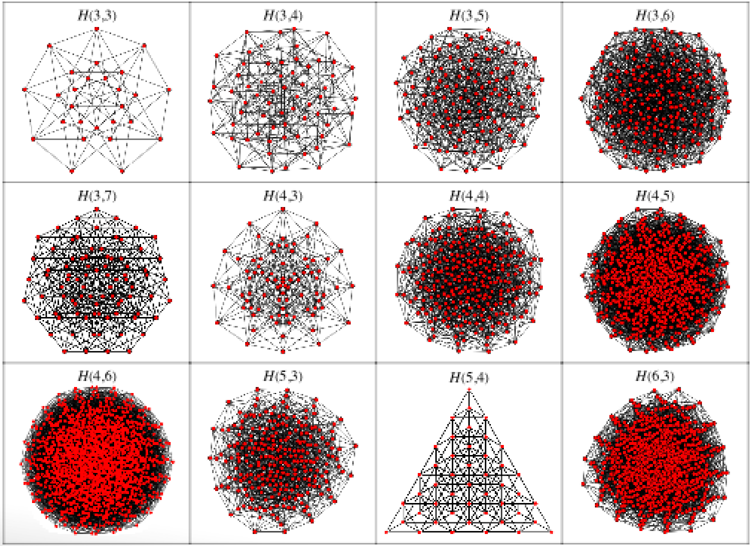
Hamming distance

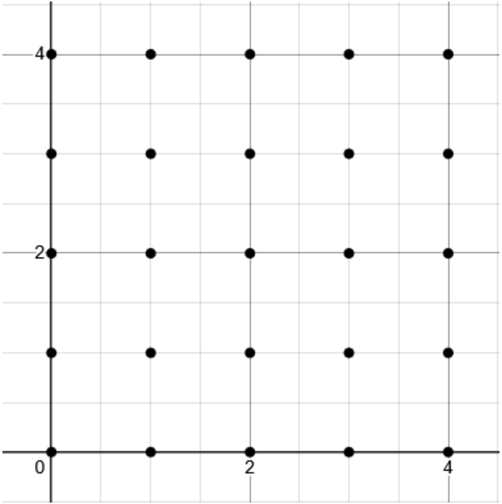
Given two points $x, y \in \mathbb{Z}_q^d$, their *Hamming distance,* $|x - y|_H$, is the number of coordinates in which they differ. That is

$$|x - y|_H := |\{j : x_j \neq y_j\}|.$$

Hamming graphs



Hamming graphs



Hamming graph

The Hamming graph

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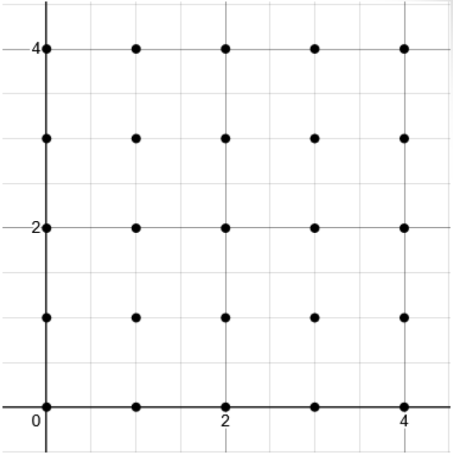

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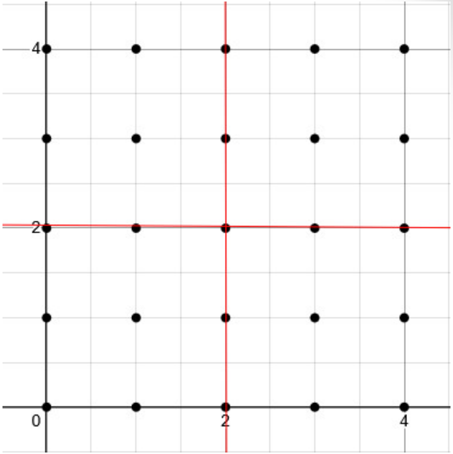

Examples

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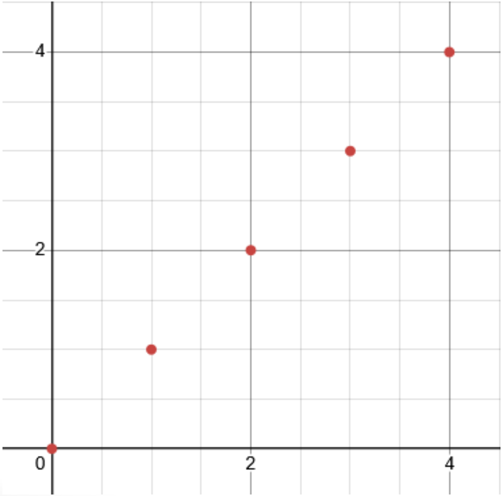
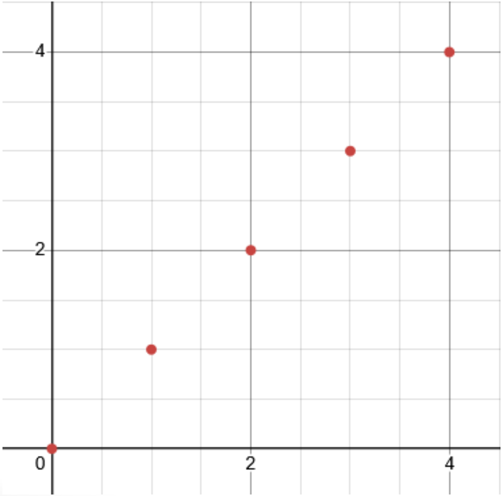


Examples

Example 1

VC-dimension 0.
All: None



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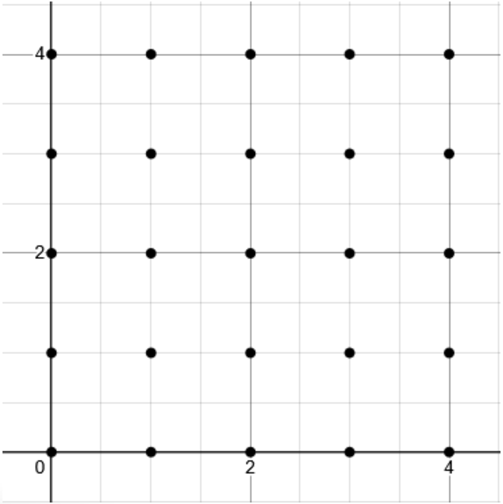
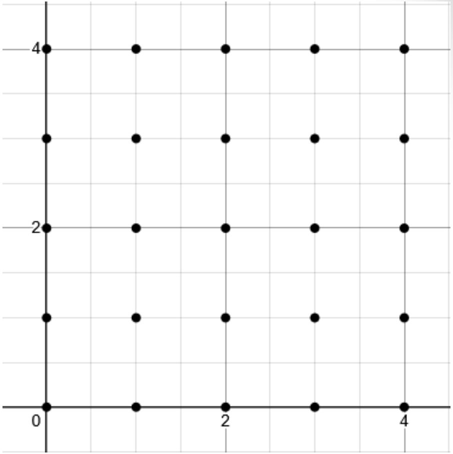


Examples



Necessary information

A Hamming graph with Hamming adjacency defined by a Hamming distance of t=1 can never have VC-dimension 4 or higher.

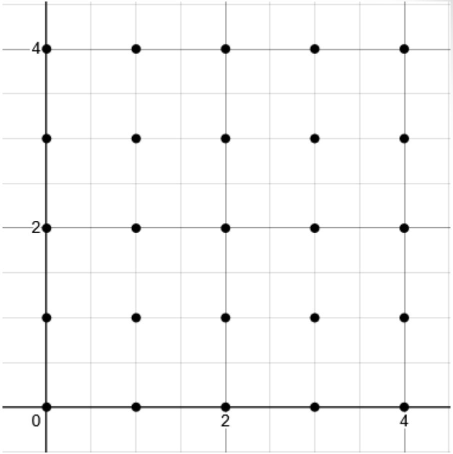
Examples

Necessary information

A Hamming graph with Hamming adjacency defined by a Hamming distance of t=1 can never have VC-dimension 4 or higher.

Why?

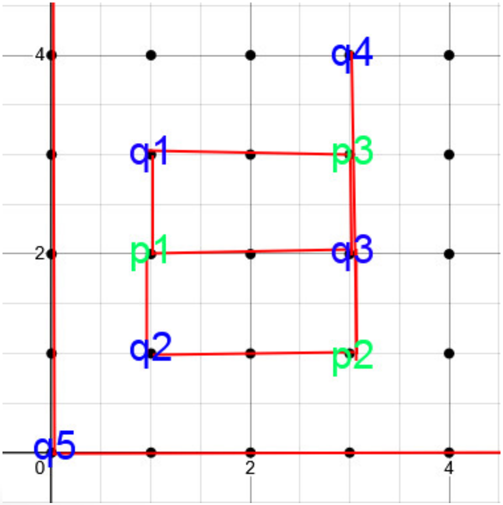
The idea is relatively easy and fun, and fun things should be left to the audience.



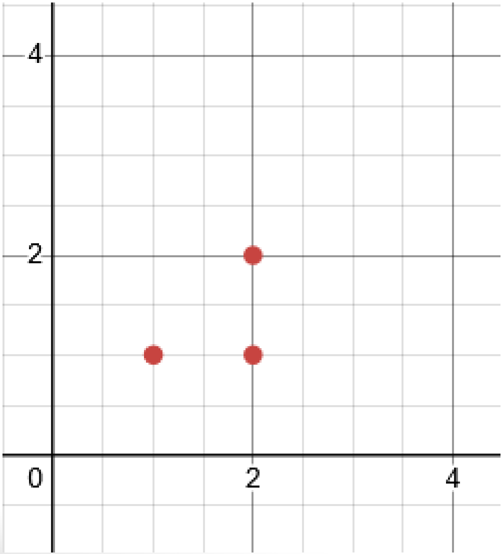
Examples

Example 2

VC-dimension 3.
q1: p1, p3
q2: p1, p2
q3: p1, p2, p3
q4: p2, p3
q5: None



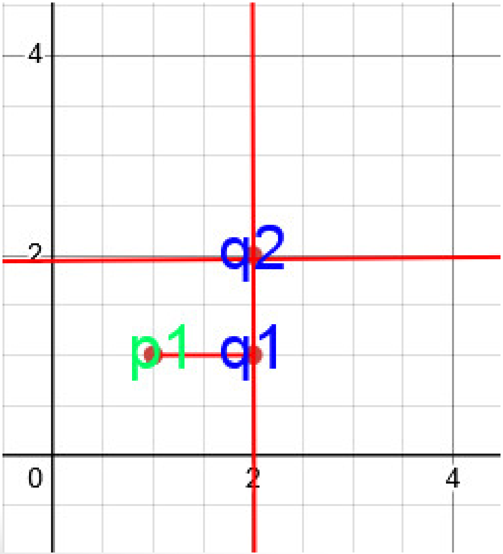
Examples



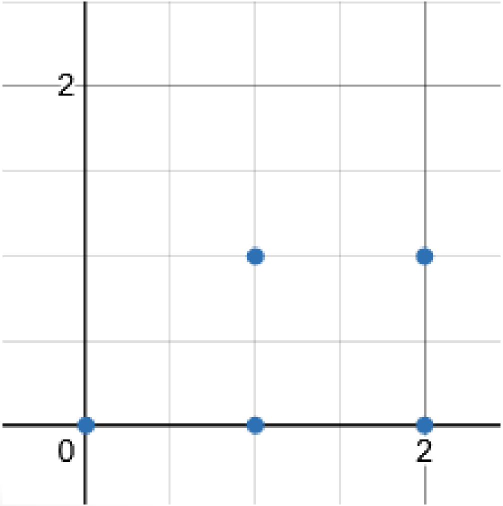
Examples

Example 3

VC-dimension 1.
q1: p1
q2: None



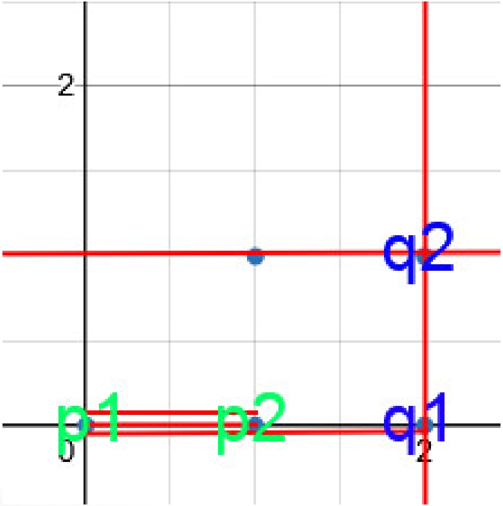
Examples



Examples

Example 4

VC-dimension 2.
q1: p1, p2
p1: p2
p2: p1
q2: None



Results

H(2, q, 1)

Theorem 0.1

Given natural numbers $d, q \geq 2$, there exists a subset U of the vertices of $H(d, q)$ of size $|U| = q$ so that the VC-dimension of $(U, n(U))$ is 0.

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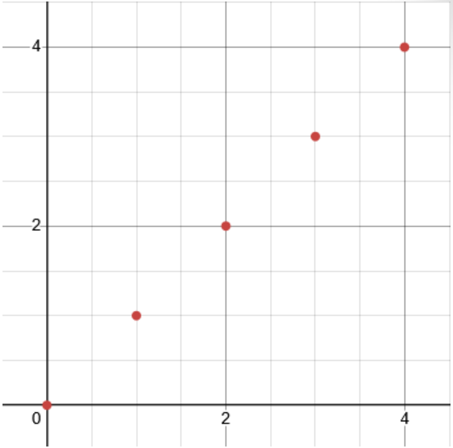

H(2, q, 1)

Theorem 1.1

Given a natural number $q \geq 2$, and subset U of the vertices of $H(2, q)$ of size $q + 1 \leq |U| < 2q$, the VC-dimension of $(U, n(U))$ is at least 1.

H(2, q, 1)

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Theorem 1.2

Given a natural number $q \geq 3$, and subset U of the vertices of $H(2,q)$ of size $|U| \geq 2q+1$, the VC-dimension of $(U, n(U))$ is at least 2.

H(2, q, 1)

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Given a natural number $q \geq 3$, and subset U of the vertices of $H(2, q)$ of size $|U| \geq 2q + 1$, the VC-dimension of $(U, n(U))$ is at least 2.

Theorem 1.3

Wait, what about when |U| = 2q???

H(2, q, 1)

Theorem 1.3.1

Given an odd number $q \geq 3$, and subset U of the vertices of $H(2, q)$ of size $|U| \geq 2q$, the VC-dimension of $(U, n(U))$ is at least 2.

Theorem 1.3.2

Given an even number $q \geq 3$, and subset U of the vertices of $H(2, q)$ of size $|U| \geq 2q + 1$, the VC-dimension of $(U, n(U))$ is at least 2.

Theorem 1.3.2 - "Proof"

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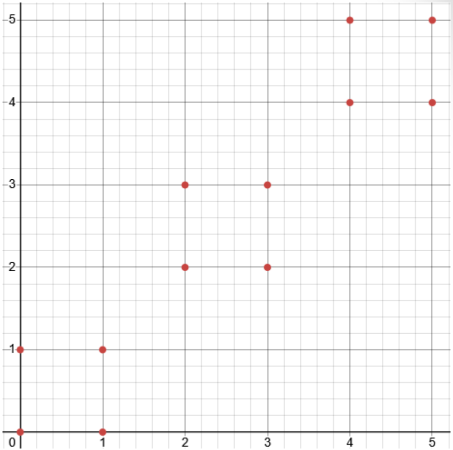

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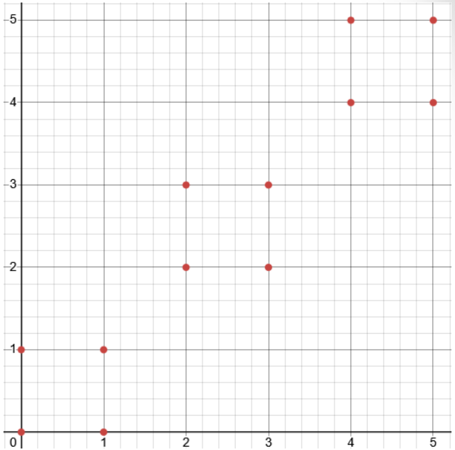
Necessary information

Given a natural number $q \geq 3$, and subset U of the vertices of $H(2,q)$ if U contains three points on the same line L, as well as a fourth point not on L, then the VC-dimension of $(U, n(U))$ is at least 2.



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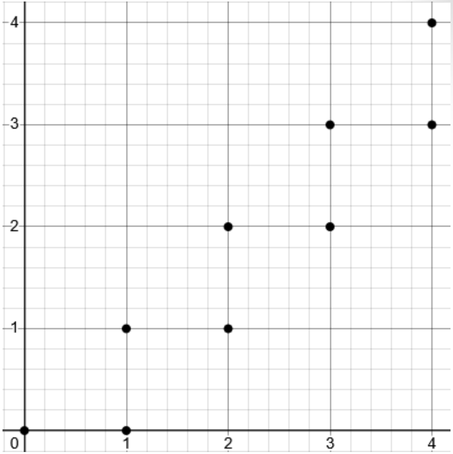

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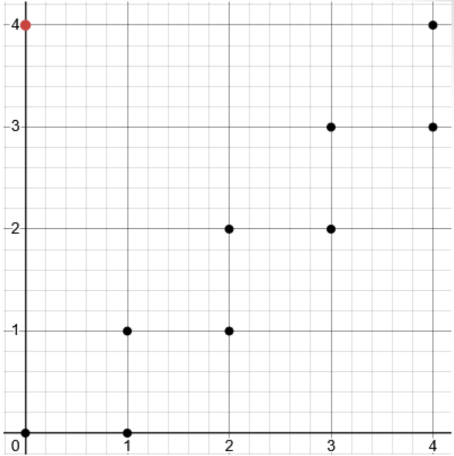

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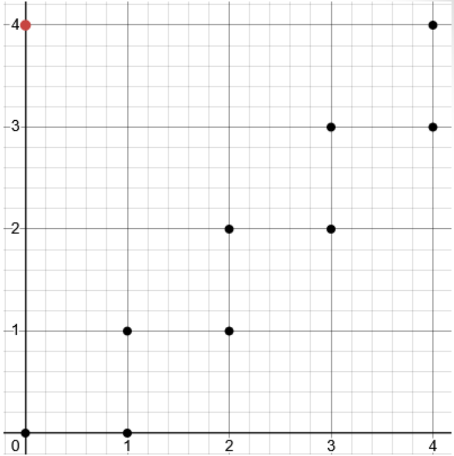
Given a natural number $q \geq 3$, and subset $|U| \geq 2q$ of the vertices of $H(2, q)$ if U contains a "corner" then the VC-dimension of $(U, n(U))$ is at least 2.



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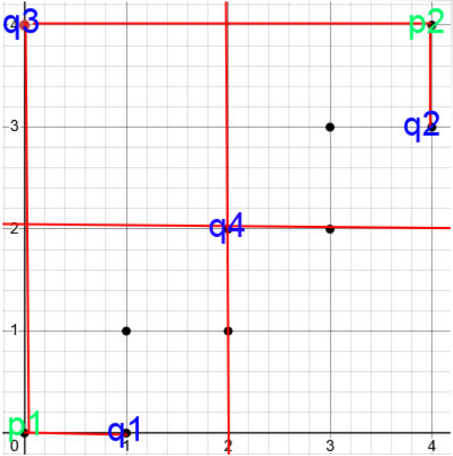

H(2, q, 1)

Theorem 1.4

For $q \geq 4$, any subset U of the vertices of $H(2,q)$ with $|U| \geq 3q+1$, the VC-dimension of $(U, n(U))$ is three.

Theorem 1.4 - Proof idea

Theorem 1.4

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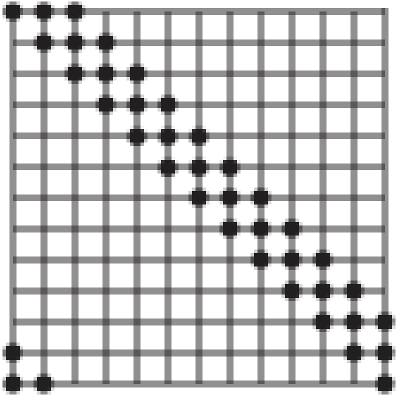
Necessary information

This is hard...



H(d, q, 1)

Theorem 2.1

Given a natural number $q \geq 3$, and subset U of the vertices of $H(d, q)$ of size $|U| \geq 2q^{d-1} + 1$, the VC-dimension of $(U, n(U))$ is at least 2.

Theorem 2.2

Given natural numbers $d, q \geq 4$, and a subset of U of the vertices of $H(d, q)$ of size $|U| \geq 3q^{d-1} + 1$, then the VC-dimension of $(U, n(U))$ is 3.

Theorem 2.1/2.2

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Necessary information

We have constructed explicit examples in $H(3, q)$ with size $\frac{5q^2}{4}$ that maintain VC-dimension 1 by avoiding all shattering pairs.

Theorem 2.1/2.2

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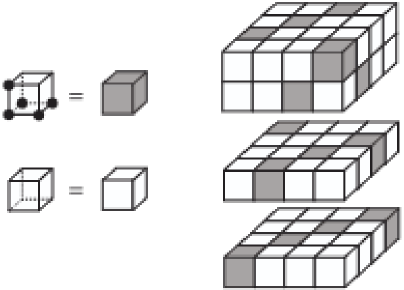

H(2, q, t)

Cool Configuration

Previously it was/we believed that VC-dimension of 4 or greater was impossible to achieve on H(2,q,t) and while this is true for H(2,q,1), we found that it is surprisingly easy to construct graphs with VC-dimension n by increasing the Hamming distance with n.

H(d,q,t)

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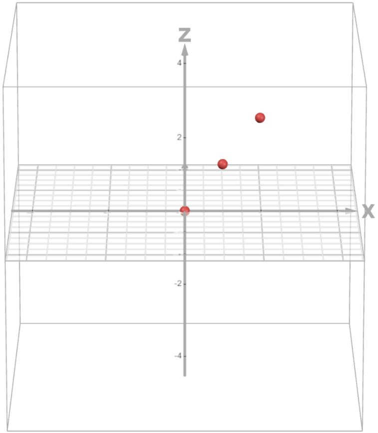
Previously it was/we believed that VC-dimension of 4 or greater was impossible to achieve on H(2,q,t) and while this is true for H(2,q,1), we found that it is surprisingly easy to construct graphs with VC-dimension n by increasing the Hamming distance with n. (i.e. t=d=n)



Thoughts, comments, concerns, critiques?

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Thanks :)

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