Bernoulli Numbers and Some Applications

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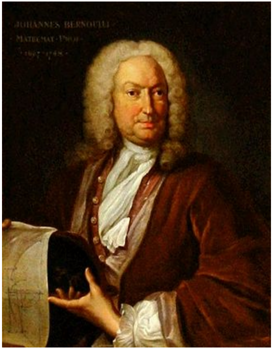
Bernoulli Family: Well-known Mathematicians

Jacob (1655 – 1705)



Johann (1667 – 1748)

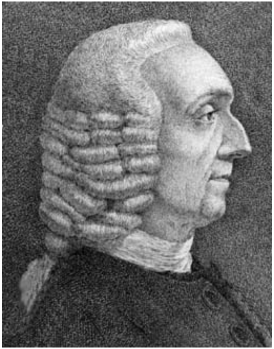
Daniel (1700 – 1782),
Son of Johann

The family relocated from Belgium to Basel, Switzerland around 1620. Jacob started as theology major and Johann started as business major. Later transferred to math, and both became math professor at Basel University.

Bernoulli Family: Some less Well-known Mathematicians

Nicolaus I (1687 – 1759),
Nephew of Jacob & Johann

Nicolaus II (1695 – 1726),
Son of Johann



Johann II (1710 – 1790),
Son of Johann

Johann III (1744 – 1807) and Jacob II (1759 – 1789), two sons of Johann II, were also mathematicians.

Jacob Bernoulli

Made many important contributions to mathematics, including

- Early development of calculus
- Bernoulli differential equation
- Bernoulli numbers, Bernoulli's formula, and Bernoulli polynomials (the topic of this talk)
- Contribution to probability theory (Bernoulli distribution, Law of large numbers)
- Calculus of variation (brachistochrone curve)
- ...

Sum of Powers of Natural Numbers

It is well known that

$$1+2+\cdots+(n-1)=\frac{1}{2}n^2-\frac{1}{2}n,$$

and

$$1^2+2^2+\cdots+(n-1)^2=\frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n.$$

Question:

Is there a formula for

$$1^m+2^m+\cdots+(n-1)^m=?$$

A Conjecture

From the previous two example, we guess that

$$S_m(n) = 1^m + 2^m + \cdots + (n-1)^m = a_{m+1}n^{m+1} + a_m n^m + \cdots + a_1 n$$

For any m, the coefficients can be calculated from some special values of n. For example, for $m = 3$, $S_3(n) = a_4 n^4 + a_3 n^3 + a_2 n^2 + a_1 n$,

$$\begin{aligned}
n = 1 &\Rightarrow S_3(1) = 0 = a_4 + a_3 + a_2 + a_1 \\
n = 2 &\Rightarrow S_3(2) = 1 = 16a_4 + 8a_3 + 4a_2 + 2a_1 \\
n = 3 &\Rightarrow S_3(3) = 9 = 81a_4 + 27a_3 + 9a_2 + 3a_1 \\
n = 4 &\Rightarrow S_3(4) = 36 = 256a_4 + 64a_3 + 16a_2 + 4a_1
\end{aligned}$$

Solve the equations, obtaining $a_4 = \frac{1}{4}, a_3 = -\frac{1}{2}, a_2 = \frac{1}{4}, a_1 = 0$.

Some Examples

Bernoulli calculated the formula for $S_m(n)$ for m up to 10, and here are some examples

$$
\begin{aligned}
S_1(n) &= \frac{1}{2}n^2 && - \frac{1}{2}n \\
S_2(n) &= \frac{1}{3}n^3 && - \frac{1}{2}n^2 && + \frac{1}{6}n \\
S_3(n) &= \frac{1}{4}n^4 && - \frac{1}{2}n^3 && + \frac{1}{4}n^2 \\
S_4(n) &= \frac{1}{5}n^5 && - \frac{1}{2}n^4 && + \frac{1}{3}n^3 && - \frac{1}{30}n \\
S_5(n) &= \frac{1}{6}n^6 && - \frac{1}{2}n^5 && + \frac{5}{12}n^4 && - \frac{1}{12}n^2 \\
S_6(n) &= \frac{1}{7}n^7 && - \frac{1}{2}n^6 && + \frac{1}{2}n^5 && - \frac{1}{6}n^3 && + \frac{1}{42}n
\end{aligned}
$$

Searching for Patterns: I

Note that for $S_m(n)$, the leading term coefficient is $\frac{1}{m+1}$. Bernoulli factored out the number $\frac{1}{m+1}$, obtaining

$$
\begin{aligned}
S_1(n) &= \frac{1}{2}\left[n^2 - n\right] \\
S_2(n) &= \frac{1}{3}\left[n^3 - \frac{3}{2}n^2 + \frac{1}{2}n\right] \\
S_3(n) &= \frac{1}{4}\left[n^4 - 2n^3 + n^2\right] \\
S_4(n) &= \frac{1}{5}\left[n^5 - \frac{5}{2}n^4 + \frac{5}{3}n^3 - \frac{1}{6}n\right] \\
S_5(n) &= \frac{1}{6}\left[n^6 - 3n^5 + \frac{5}{2}n^4 - \frac{1}{2}n^2\right] \\
S_6(n) &= \frac{1}{7}\left[n^7 - \frac{7}{2}n^6 + \frac{7}{2}n^5 - \frac{7}{6}n^3 + \frac{1}{6}n\right]
\end{aligned}
$$

Searching for Patterns: II

The next observation is that

- Each of the coefficient of the second column term is a multiple of $-\frac{1}{2}$
- Each of the coefficient of the third column term is a multiple of $\frac{1}{6}$
- Each of the coefficient of the fourth column term is a multiple of $-\frac{1}{30}$
- ...

Searching for Patterns: III

We can rewrite the last page formula as

$$
\begin{aligned}
S_1(n) &= \frac{1}{2}\left[1n^2 + 2\left(-\frac{1}{2}\right)n\right]\\
S_2(n) &= \frac{1}{3}\left[1n^3 + 3\left(-\frac{1}{2}\right)n^2 + 3\left(\frac{1}{6}\right)n\right]\\
S_3(n) &= \frac{1}{4}\left[1n^4 + 4\left(-\frac{1}{2}\right)n^3 + 6\left(\frac{1}{6}\right)n^2\right]\\
S_4(n) &= \frac{1}{5}\left[1n^5 + 5\left(-\frac{1}{2}\right)n^4 + 10\left(\frac{1}{6}\right)n^3 + 5\left(-\frac{1}{30}\right)n\right]\\
S_5(n) &= \frac{1}{6}\left[1n^6 + 6\left(-\frac{1}{2}\right)n^5 + 15\left(\frac{1}{6}\right)n^4 + 15\left(-\frac{1}{30}\right)n^2\right]\\
S_6(n) &= \frac{1}{7}\left[1n^7 + 7\left(-\frac{1}{2}\right)n^6 + 21\left(\frac{1}{6}\right)n^5 + 35\left(-\frac{1}{30}\right)n^3 + 7\left(\frac{1}{42}\right)n\right]
\end{aligned}
$$

Searching for Patterns: IV

Let us fill in the missing powers

$$S_1(n) = \frac{1}{2}\left[1\cdot 1\cdot n^2 + 2\left(-\frac{1}{2}\right)n\right]$$

$$S_2(n) = \frac{1}{3}\left[1\cdot 1\cdot n^3 + 3\left(-\frac{1}{2}\right)n^2 + 3\left(\frac{1}{6}\right)n\right]$$

$$S_3(n) = \frac{1}{4}\left[1\cdot 1\cdot n^4 + 4\left(-\frac{1}{2}\right)n^3 + 6\left(\frac{1}{6}\right)n^2 + b\cdot 0\cdot n\right]$$

$$S_4(n) = \frac{1}{5}\left[1\cdot 1\cdot n^5 + 5\left(-\frac{1}{2}\right)n^4 + 10\left(\frac{1}{6}\right)n^3 + b\cdot 0\cdot n^2 + 5\left(-\frac{1}{30}\right)n\right]$$

$$S_5(n) = \frac{1}{6}\left[1\cdot 1\cdot n^6 + 6\left(-\frac{1}{2}\right)n^5 + 15\left(\frac{1}{6}\right)n^4 + b\cdot 0\cdot n^3 + 15\left(-\frac{1}{30}\right)n^2 + b\cdot 0\cdot n\right]$$

$$S_6(n) = \frac{1}{7}\left[1\cdot 1\cdot n^7 + 7\left(-\frac{1}{2}\right)n^6 + 21\left(\frac{1}{6}\right)n^5 + b\cdot 0\cdot n^4 + 35\left(-\frac{1}{30}\right)n^3 + b\cdot 0\cdot n^2 + 7\left(\frac{1}{42}\right)n\right]$$

What are the integers?

Note the similarity between the red numbers and Pascal triangle:

$$\begin{array}{cccccccc}
1 & & & & & & & \\
1 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
1 & 3 & 3 & 1 & & & & \\
1 & 4 & 6 & 4 & 1 & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1
\end{array}$$

Hence, the red numbers in the formula last page are just the elements of Pascal triangles. Since the blue number (b) can be any number, we can take the coefficients in the previous formula as Pascal numbers, that is, combination numbers!

Searching for Patterns: V

With this new insight, we rewrite the formulas as

$$S_1(n) = \frac{1}{2}\left[\binom{2}{0}\cdot 1\cdot n^2 + \binom{2}{1}\left(-\frac{1}{2}\right)n\right]$$

$$S_2(n) = \frac{1}{3}\left[\binom{3}{0}\cdot 1\cdot n^3 + \binom{3}{1}\left(-\frac{1}{2}\right)n^2 + \binom{3}{2}\left(\frac{1}{6}\right)n\right]$$

$$S_3(n) = \frac{1}{4}\left[\binom{4}{0}\cdot 1\cdot n^4 + \binom{4}{1}\left(-\frac{1}{2}\right)n^3 + \binom{4}{2}\left(\frac{1}{6}\right)n^2 + \binom{4}{3}\cdot 0\cdot n\right]$$

$$S_4(n) = \frac{1}{5}\left[\binom{5}{0}\cdot 1\cdot n^5 + \binom{5}{1}\left(-\frac{1}{2}\right)n^4 + \binom{5}{2}\left(\frac{1}{6}\right)n^3 + \binom{5}{3}\cdot 0\cdot n^2 + \binom{5}{4}\left(-\frac{1}{30}\right)n\right]$$

$$S_5(n) = \frac{1}{6}\left[\binom{6}{0}\cdot 1\cdot n^6 + \binom{6}{1}\left(-\frac{1}{2}\right)n^5 + \binom{6}{2}\left(\frac{1}{6}\right)n^4 + \binom{6}{3}\cdot 0\cdot n^3 + \binom{6}{4}\left(-\frac{1}{30}\right)n^2 + \binom{6}{5}\cdot 0\cdot n\right]$$

$$S_6(n) = \frac{1}{7}\left[\binom{7}{0}\cdot 1\cdot n^7 + \binom{7}{1}\left(-\frac{1}{2}\right)n^6 + \binom{7}{2}\left(\frac{1}{6}\right)n^5 + \binom{7}{3}\cdot 0\cdot n^4 + \binom{7}{4}\left(-\frac{1}{30}\right)n^3 + \binom{7}{5}\cdot 0\cdot n^2 + \binom{7}{6}\left(\frac{1}{42}\right)n\right]$$

The mysterious numbers in the black parentheses are called Bernoulli numbers, denoted as B_k. In particular,

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0,$$

and

$$B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, B_7 = 0, \cdots$$

The Formula!!!

Hence,

$$\begin{aligned} S_6(n) &= \frac{1}{7}\left[\binom{7}{0}B_0n^7+\binom{7}{1}B_1n^6+\binom{7}{2}B_2n^5+\binom{7}{3}B_3n^4+\binom{7}{4}B_4n^3+\binom{7}{5}B_5n^2+\binom{7}{6}B_6n\right] \\ &= \frac{1}{7}\sum_{j=0}^{6}\binom{7}{j}B_jn^{7-i} = \frac{1}{6+1}\sum_{j=0}^{6}\binom{6+1}{j}B_jn^{6+1-j} \end{aligned}$$

In general, the long-sought-after formula for sum of powers is

$$\begin{aligned} S_k(n) &= 1^k+2^k+3^k+\cdots+(n-1)^k \\ &= \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_jn^{k+1-j}. \end{aligned}$$

This is called Bernoulli formula! Although summarized from heuristic and observation, this formula could be proved (later for an idea).

Formal Definition

In 1755, L. Euler defined Bernoulli number in the following modern way

The Bernoulli numbers are the coefficients of exponential generating function of

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!}$$

If we let

$$f(x) = \frac{x}{e^x - 1},$$

we have

$$B_n = \left. \frac{d^n f(x)}{dx^n} \right|_{x=0}.$$

Leonhard Euler (1707 – 1783)

- Student of Johann Bernoulli, friend of Daniel and Nicolaus Bernoulli (sons of Johann), started as theology major.
- Pioneered **graph theory**, topology, and modern mathematical notation
- Advanced **analysis** and **number theory** with lasting formulas and functions
- Made fundamental contributions to **mechanics, astronomy, and optics**
- Beethoven in mathematics! Lost right eye sight at 31, totally blind around 59. Some sources say he produced almost half his total works despite the total blindness.
- Terms named after him, too many to list ....

Sum of Powers of Natural Numbers: Revisit

Consider the generating function of $S_k(n) = \sum_{m=0}^{n-1} m^k$,

$$
\begin{aligned}
S_n(t) &= \sum_{k=0}^{\infty} S_k(n) \frac{t^k}{k!} = \sum_{k=0}^{\infty} \sum_{m=0}^{n-1} m^k \frac{t^k}{k!} = \sum_{m=0}^{n-1} \sum_{k=0}^{\infty} \frac{(mt)^k}{k!} = \sum_{m=0}^{n-1} e^{mt} \\
&= \frac{e^{nt}-1}{e^t-1} = \frac{e^{nt}-1}{t} \frac{t}{e^t-1} = \sum_{i=1}^{\infty} \frac{(nt)^i}{i!t} \sum_{j=0}^{\infty} \frac{B_j t^j}{j!} = \sum_{i=0}^{\infty} \frac{n^{i+1}}{i+1} \frac{t^i}{i!} \sum_{j=0}^{\infty} \frac{B_j t^j}{j!} \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{n^{k-j+1}}{k-j+1} \frac{t^{k-j}}{(k-j)!} \frac{B_j t^j}{j!} = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} \binom{k}{j} \frac{B_j n^{k+1-j}}{k+1-j} \right] \frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} \right] \frac{t^k}{k!}.
\end{aligned}
$$

Comparing the coefficients at the two ends, we have

$$
S_k(n) = \sum_{m=0}^{n-1} m^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j}.
$$

Some Examples of Bernoulli Numbers

We can calculate

$$f(x) = \frac{x}{e^x - 1}, \quad \text{and} \quad B_0 = \lim_{x \to 0} f(x) = 1,$$

$$f'(x) = \frac{xe^x - e^x + 1}{(e^x - 1)^2}, \quad \text{and} \quad B_1 = \lim_{x \to 0} f'(x) = -\frac{1}{2},$$

and

$$f''(x) = \frac{xe^{2x} - xe^x - 2e^{2x} + 2e^x}{(e^x - 1)^3}, \quad \text{and} \quad B_2 = \lim_{x \to 0} f''(x) = \frac{1}{6}.$$

Some Bernoulli Numbers

$B_0 = 1$

$B_1 = -1/2$

$B_2 = 1/6$

$B_3 = 0$

$B_4 = -1/30$

$B_5 = 0$

$B_6 = 1/42$

$B_7 = 0$

$B_8 = -1/30$

$B_9 = 0$

$B_{10} = 5/66$

$B_{11} = 0$

$B_{12} = -691/2730$

$B_{13} = 0$

$B_{14} = 7/6$

$B_{15} = 0$

$B_{16} = -3617/510$

$B_{17} = 0$

$B_{18} = 43867/798$

$\cdots$

$B_{49} = 0$

$B_{50} = 495057205241079648212477525/66$

Basic Properties of Bernoulli Numbers

- B_n is rational for any $n \geq 1$. [This could be proved using a recursive relation of the Bernoulli numbers, next slide.]

- $B_{2n+1} = 0$ for $n \geq 1$. [$f(x) - B_1 x$ is even function, see later.]

- B_{2n} alternates signs in the sense $B_{4n} < 0$ and $B_{4n+2} > 0$ for $n \geq 1$. [Using the relationship between B_{2n} and $\zeta(2n)$, later.]

- The magnitude of B_{2n} grows very quickly. [Using the relationship between B_{2n} and $\zeta(2n)$, later.]

Bernoulli Numbers are Rational

By definition

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!},$$

we have

$$x = (e^x - 1) \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} = \sum_{j=1}^{\infty} \frac{x^j}{j!} \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} = \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \binom{n}{k} B_k \frac{x^n}{n!}.$$

Comparing coefficients on both sides, there is

$$B_0 = 1, \text{ and } \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \text{ for } n > 1.$$

From this recursive relation, we can show by induction that all B_n are rational numbers.

Power Series for Sine and Cosine

It is well known from calculus that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

Question: what about other trigonometric functions?

Bernoulli number and cotangent: I

Consider function

$$f(x) - B_1 x = \frac{x}{e^x - 1} - B_1 x = \frac{x(e^{x/2} + e^{-x/2})}{2(e^{x/2} - e^{-x/2})} = \frac{x}{2} \frac{\frac{1}{2}(e^{x/2} + e^{-x/2})}{\frac{1}{2}(e^{x/2} - e^{-x/2})}$$

$$= \frac{x}{2} \frac{\cosh(x/2)}{\sinh(x/2)} = \frac{x}{2} \coth \frac{x}{2}. \quad \text{Even function!}$$

On the other hand, from the definition of Bernoulli number, we have

$$f(x) - B_1 x = \frac{x}{e^x - 1} - B_1 x = \sum_{k=0}^{\infty} \frac{B_k x^k}{k!} - B_1 x = \sum_{k=0, k \neq 1}^{\infty} \frac{B_k x^k}{k!}$$

$$(\text{Even, so } B_{\text{odd}} = 0) = \sum_{n=0}^{\infty} \frac{B_{2n} x^{2n}}{(2n)!}.$$

Bernoulli number and cotangent: II

Hence

$$\frac{x}{2}\coth\frac{x}{2}=\sum_{n=0}^{\infty}\frac{B_{2n}x^{2n}}{(2n)!} \qquad \text{or} \quad x\coth x=\sum_{n=0}^{\infty}\frac{B_{2n}(2x)^{2n}}{(2n)!}$$

Since $\coth(ix)=-i\cot x$, we can find that

$$ix\coth(ix)=\sum_{n=0}^{\infty}\frac{B_{2n}(2ix)^{2n}}{(2n)!} \quad \text{or} \quad ix(-i\cot x)=\sum_{n=0}^{\infty}(-1)^{n}\frac{B_{2n}(2x)^{2n}}{(2n)!}$$

Finally,

$$\cot x=2\sum_{n=0}^{\infty}(-1)^{n}\frac{B_{2n}(2x)^{2n-1}}{(2n)!}$$

with the first few terms

$$\cot x=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}+\frac{x^7}{4725}+\cdots$$

Power series for tanh and tan

We observe

$$2\coth(2z) - \coth(z) = 2\frac{\cosh(2z)}{\sinh(2z)} - \frac{\cosh(z)}{\sinh(z)}$$
$$= \frac{\cosh^2(z) + \sinh^2(z)}{\sinh(z)\cosh(z)} - \frac{\cosh(z)}{\sinh(z)} = \tanh(z)$$

A little bit more work shows that

$$\tanh(x) = \sum_{n=1}^{\infty} 4^n(4^n - 1)\frac{B_{2n}x^{2n-1}}{(2n)!}$$

and

$$\tan(x) = \sum_{n=1}^{\infty} (-4)^n(1 - 4^n)\frac{B_{2n}x^{2n-1}}{(2n)!}.$$

with the first few terms

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

Power Series for $\csc x$

- First, verify that
$$\csc(x) = \cot\frac{x}{2} - \cot x.$$
- Use the expression for $\cot x$ before, rearrange the terms, to get
$$\csc(x) = 2\sum_{n=0}^{\infty}(-1)^{n-1}(2^{2n-1}-1)\frac{B_{2n}x^{2n-1}}{(2n)!}.$$

  with the first few terms
$$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots$$

Finally, $\sec x$

- By introducing another type of number called Euler number E_n, we have

$$\sec(x) = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}.$$

with the first few terms

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

- Using the relationship between trigonometric functions and hyperbolic functions, we can get the power series for sech and csch

Bernoulli numbers and the Riemann Zeta Function

We know that the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for} \quad Re(s) > 1$$

In 1734, Euler found

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \qquad \text{Basel problem.}$$

Using the power series for $\cot x$ and the infinite product formula for $\sin x$, and a big amount of effort, we can get

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n} B_{2n}}{2(2n)!} = \frac{|B_{2n}|(2\pi)^{2n}}{2(2n)!}$$

Sign of B_{even}

From the above formula,

$$B_{2n} = (-1)^{n+1} \frac{2\zeta(2n)(2n)!}{(2\pi)^{2n}}.$$

- When n is even, i.e. $n = 2k$,

$$\operatorname{sign}(B_{2n}) = \operatorname{sign}(B_{4k}) = \operatorname{sign}((-1)^{2k+1}) = -1;$$

- When n is odd, i.e. $n = 2k+1$,

$$\operatorname{sign}(B_{2n}) = \operatorname{sign}(B_{4k+2}) = \operatorname{sign}((-1)^{2k+1+1}) = 1.$$

Approximate Bernoulli numbers

When n is large, we have

$$\zeta(2n) = 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots \approx 1.$$

Thus,

$$B_{2n} \approx (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}}.$$

Using Stirling's formula, we have

$$(2n)! \approx \sqrt{4n\pi} \left(\frac{2n}{e} \right)^{2n}.$$

Hence

$$B_{2n} \approx (-1)^{n-1} \frac{2}{(2\pi)^{2n}} \sqrt{4n\pi} \left(\frac{2n}{e} \right)^{2n} = 4(-1)^{n-1} \sqrt{n\pi} \left(\frac{n}{e\pi} \right)^{2n}$$

Tail of ζ-number

Let

$$t_s(n) = \sum_{k=1}^{\infty} \frac{1}{(n+k)^s} = \zeta(s) - \left[\frac{1}{1^s} + \frac{1}{2^s} + \cdots + \frac{1}{n^s}\right].$$

Notice that

$$\frac{1}{(n+k)^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-(n+k)x} dx.$$

Hence, we have

$$\begin{aligned} t_s(n) &= \frac{1}{\Gamma(s)} \sum_{k=1}^{\infty} \int_0^\infty x^{s-1} e^{-(n+k)x} dx = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-nx} \sum_{k=1}^{\infty} e^{-kx} dx \\ &= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-nx} \frac{e^{-x}}{1-e^{-x}} dx = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-2} e^{-nx} \frac{x}{e^x - 1} dx \\ &= \frac{1}{\Gamma(s)} \int_0^\infty x^{s-2} e^{-nx} \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k dx = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{B_k}{k!} \int_0^\infty x^{s+k-2} e^{-nx} dx \\ &= \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{B_k}{k!} \frac{1}{n^{s+k-1}} \int_0^\infty y^{s+k-2} e^{-y} dy = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{B_k}{k!} \frac{\Gamma(s+k-1)}{n^{s+k-1}}. \end{aligned}$$

Tail of $\zeta(2)$ and $\zeta(3)$

Specifically, for $\zeta(2)$,

$$\begin{aligned} t_2(n) &= \frac{1}{\Gamma(2)} \sum_{k=0}^{\infty} \frac{B_k}{k!} \frac{\Gamma(k+1)}{n^{k+1}} = \sum_{k=0}^{\infty} \frac{B_k}{n^{k+1}} \\ &= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{6n^3} + \cdots \\ &\approx \frac{1}{n}. \end{aligned}$$

and for $\zeta(3)$,

$$\begin{aligned} t_3(n) &= \frac{1}{\Gamma(3)} \sum_{k=0}^{\infty} \frac{B_k}{k!} \frac{\Gamma(k+2)}{n^{k+2}} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(k+1)B_k}{n^{k+2}} \\ &= \frac{1}{2} \left(\frac{1}{n^2} - \frac{1}{n^3} + \frac{1}{2n^4} + \cdots \right) \\ &\approx \frac{1}{2n^2}. \end{aligned}$$

Thank You!

Any Question?