Simplicial Monomial Rings

There are many properties a commutative ring may possess, for example it may be Cohen-Macaulay, Gorenstein, or Buchsbaum. Normally, these concepts would be too advanced for an undergraduate to do research on. However, if we restrict ourselves to simplicial monomial rings, these properties can be expressed in terms of the number-theoretical and combinatorial properties of the underlying semigroup [1,5]. This transforms many problems that might not be easily accessible to undergraduates to ones that are. For example, the author's coauthor Leslie Roberts has conducted a successful undergraduate research project with Victoria De Quehen (with extensive use of *Mathematica*) that found a counterexample to a conjecture about Cohen-Macaulay projective monomial curves [2].

The author has shown that the density of Cohen-Macaulay projective monomial curves (and hence Gorenstein curves) approaches zero as the degree goes to infinity [3]. Subsequent work [4] has shown that, letting CM_d denote the number of projective monomial curves that are Cohen-Macaulay, $\liminf_{d\to\infty} CM_d^{1/d} \geq \sqrt{2} \approx 1.41421$, and $\limsup_{d\to\infty} CM_d^{1/d} \leq 1.87384$. It would be of great interest to show that $\lim_{d\to\infty} CM_d{}^{1/d}$ exists and to evaluate it or, barring that, to improve on the estimates above.

Numerical evidence indicates that the density of Buchsbaum projective monomial curves should also approach zero as the degree grows arbitrarily large. One would like to prove this and to study the asymptotics of this behavior, as well as the asymptotic behavior of Gorenstein projective curves.

All of the questions above may also be asked about higher-dimensional simplicial monomial rings.

The combinatorial/number-theoretic nature of this project makes it readily accessible to undergraduates. The fact that monomial rings can be represented graphically (at least in the two- and three-dimensional cases) should make this investigation particularly attractive to undergraduates. Upon completion of the project, the participant should have acquired a solid grounding in commutative algebra and an increased depth of knowledge in number theory and/or combinatorics.

Prerequisites: One semester of abstract algebra. Some elementary number theory is desirable, but not required.

References

[1] W. Bruns, J. Gubeladze, and N.V. Trung, *Problems and algorithms for affine semigroups*, Forum **64** (2002), 180-212.

[2] V. De Quehen and L. Roberts, *Non-Cohen-Macaulay projective monomial curves with positive h-vector*, Canadian Mathematical Bulletin **48** 2005, 203-210.

[3] L. Reid and L. Roberts, *Non-Cohen-Macaulay projective monomial curves*, Journal of Algebra, **291** (2005), 171-186.

[4] L. Reid and L. Roberts, *Maximal and Cohen-Macaulay curves*, Journal of Algebra, to appear.

[5] N.V. Trung, and L.T. Hoa, *Affine semigroups and Cohen-Macaulay rings generated by monomials*, Transactions of the AMS **298** (1986), 145-167.